Ch.2. Shallow Water Rossby Wave Dynamics

Sec. 2.1: Quasi-Geostrophic Equation

1. Nondimensional Equations

To focus on low frequency variability, we would like to filter out the high frequency modes in the shallow water equations. Mathematically, it will be convenient if we can have a single equation in a single variable to govern the variability of large-scale flows. The derivation of a simpler set of equations for specific temporal and spatial scales is accomplished usually with perturbation method.

We first nondimensionalize the shallow water equations. Denoting dimensional variables with an *, we have

$$(u^*, v^*) = U(u, v), \quad \eta^* = D + N\eta, \quad z_B^* = Nz_B, \quad (x^*, y^*) = L(x, y), \quad t^* = Tt,$$

where u, v, z_B, η, x, y, t are all $O(1)$ dimensionless variables. The shallow water equations can be written in the dimensionless variables as:

$$\begin{cases} \dfrac{U}{T}\partial_t u + \dfrac{U^2}{L}(u\partial_x u + v\partial_y u) - fUv = -g\dfrac{N}{L}\partial_x \eta + \dfrac{1}{\rho_0}F_x \\ \dfrac{U}{T}\partial_t v + \dfrac{U^2}{L}(u\partial_x v + v\partial_y v) + fUu = -g\dfrac{N}{L}\partial_y \eta + \dfrac{1}{\rho_0}F_y \\ \dfrac{N}{T}\partial_t \eta + \dfrac{UN}{L}\left[u\partial_x(\eta - z_B) + v\partial_y(\eta - z_B)\right] + \dfrac{U}{L}\left[D + N(\eta - z_B)\right]\left[\partial_x u + \partial_y v\right] = 0 \end{cases}$$

Assuming a beta-plane $f = f_0 + \beta Ly = f_0(1 + \dfrac{\beta L}{f_0}y)$ and dividing the u- and v-equations by f_0U, we have the non-dimensional momentum equations

$$\varepsilon_T\partial_t u + \varepsilon(u\partial_x u + v\partial_y u) - (1 + by)v = -\frac{gN}{f_0UL}\partial_x\eta + G_x$$

$$\varepsilon_T\partial_t v + \varepsilon(u\partial_x v + v\partial_y v) + (1 + by)u = -\frac{gN}{f_0UL}\partial_y\eta + G_y$$

where $\varepsilon_T = \dfrac{1}{f_0T}$, $\varepsilon = \dfrac{U}{f_0L}$, $b = \dfrac{\beta L}{f_0}$ are dimensionless parameters.

Dividing the mass equation by UD/L, we have the nondimensional mass equation

$$\frac{NL}{TUD}\partial_t\eta + \frac{N}{D}\left[u\partial_x(\eta - z_B) + v\partial_y(\eta - z_B)\right] + \left[1 + \frac{N}{D}(\eta - z_B)\right](\partial_x u + \partial_y v) = 0$$

or, denote $\delta = N/D$ and in dimensionless parameters as

$$\frac{\varepsilon_T}{\varepsilon}\delta\partial_t\eta + \delta\left[u\partial_x(\eta - z_B) + v\partial_y(\eta - z_B)\right] + (1 + \delta(\eta - z_B))(\partial_x u + \partial_y v) = 0\,.$$

We are interested in the flow with:

1) a slow time scale (relative to 1/*f*) such that $\varepsilon_T << 1$ (small KnδeΩb No.),

2) a large scale or weak flow such that $\varepsilon << 1$ (small Rossby No.)

3) a weak forcing and dissipation so that $G \sim O(\varepsilon) << 1$ (or $G \sim \varepsilon E$, $E \sim O(1)$).

In the momentum equations, assumptions (1)-(3) lead to the first order balance between the pressure gradient force and the Corilois force, such that $f_0 U \sim gN/L$. This gives the scale of the pressure anomaly in terms of those of velocity and space as

$$N \sim O(f_0 UL/g)$$

Furthermore, we require that the surface elevation is small compared with the total depth. Denoting $F = \left(\frac{L}{L_D}\right)^2$ as the Froude number, we therefore have

$$\delta = \frac{N}{D} \sim \frac{f_0 UL}{gD} \sim \frac{f_0^2}{gD}\frac{U}{f_0 L} \cdot L^2 \sim \left(\frac{L}{L_D}\right)^2 \cdot \varepsilon = F\varepsilon << 1$$

or

4) $F << O(1/\varepsilon)$

Therefore, the scale can't be too much larger than the deformation radius. For synoptic processes with $L \sim L_D$, this condition is satisfied. With assumptions (1)-(4), we have

$$\delta \le O(\varepsilon) << 1.$$

The other condition on the (meridional scale) of the motion is

5) $b \sim \frac{\beta L}{f_0} \sim \frac{fL/a}{f_0} \sim \frac{L}{a} << 1$

or $b \sim \gamma\varepsilon$ with $\gamma \sim O(1)$. This requires that L can't be global ($\sim a$). Finally, we assume that the time scale is comparable with the advective time scale

6) $\varepsilon_T \sim \varepsilon$ *or* $T \sim \frac{L}{U}$.

The nondimensional equations can be written as:

$$\begin{cases} \varepsilon(\partial_t u + u\partial_x u + v\partial_y u) - (1+\gamma\varepsilon y)v = -\partial_x \eta + \varepsilon E_x \\ \varepsilon(\partial_t v + u\partial_x v + v\partial_y v) + (1+\gamma\varepsilon y)u = -\partial_y \eta + \varepsilon E_y \\ \varepsilon F\left[\partial_t h + u\partial_x(\eta - z_B) + v\partial_y(\eta - z_B)\right] + \left[1 + \varepsilon F(\eta - z_B)\right]\left(\partial_x u + \partial_y v\right) = 0 \end{cases} \quad (2.1.1a,b,c)$$

The variables are expanded as

$$\begin{cases} u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\cdots\cdot \\ v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots\cdots\cdot \\ \eta = \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \cdots\cdots\cdot \end{cases}$$

We will collect terms of the same order in (2.1.1) and derive the leading order equations.

2. O(1) Equations

Collecting terms of order O(1), we have the geostrophic balance

$$\begin{cases} -v_0 = -\partial_x \eta_0 \\ u_0 = -\partial_y \eta_0 \\ \partial_x u_0 + \partial_y v_0 = 0 \end{cases} \qquad (2.1.2a,b,c)$$

This is the same as the low frequency $\omega=0$ geostrophic mode studied in section 1.5. As discussed before, geostrophic balance is degenerated: any pressure field satisfies the equation! The deterministic part is at the next order. The balance itself is simply a self consistent diagnostic relationship.

Note 1 Why large scale tends to be geostrophic?

Consider the u-equation, $\partial_t u + u\partial_x u + v\partial_y u - fv = -g\eta_x + A\partial_{xx} u$. The only term that is independent of spatial scale (at low frequency) is the Coriolis force *fv*. Therefore, as spatial scale increases, all the terms decrease except for the Coriolis force. For large enough *L*, *fv* has to be the dominant term to balance the pressure gradient force. That is geostrophy!

Note 2 What happens for high topography?

In the above, we have assumed that the topographic height is low compared with the total depth of the fluid

$$\frac{O(z_B)}{O(D)} \sim O(\delta) \le O(\varepsilon) \le 1$$

For high topography, $\frac{O(z_B)}{O(D)} \sim 1$. The flow, instead of climbing over mountains, flows around mountains.

3. O(ε) Equations

At the next order, we have the equations

$$\begin{aligned} & -v_1 + \partial_t u_0 + u_0 \partial_x u_0 + v_0 \partial_y u_0 - \gamma y v_0 = -\partial_x \eta_1 + E_x \\ & u_1 + \partial_t v_0 + u_0 \partial_x v_0 + v_0 \partial_y u_0 + \gamma y u_0 = -\partial_y \eta_1 + E_y \qquad (2.1.3a,b,c) \\ & \partial_x u_1 + \partial_y v_1 + F(\eta_0 - z_B)(\partial_x u_0 + \partial_y v_0) + F\left[\partial_t \eta_0 + u_0 \partial_x (\eta_0 - z_B) + v_0 \partial_y (\eta_0 - z_B)\right] = 0 \end{aligned}$$

.

$\partial_x(2.1.3b) - \partial_y(2.1.3a)$ gives the vorticity equation

$$(\partial_t + u_0\partial_x + v_0\partial_y)(\partial_x v_0 - \partial_y u_0) + \partial_x u_1 + \partial_y v_1 + \gamma v_0 = \partial_x E_y - \partial_y E_x$$

Plug in (2.1.3c) to eliminate the divergence, we have the quasi-geostrophic potential vorticity equation (QGPV)

$$\boxed{(\partial_t + u_0\partial_x + v_0\partial_y)\Pi = curlE} \qquad (2.1.4)$$

where

$$\Pi = \gamma y + \partial_x v_0 - \partial_y u_0 - F(\eta_0 - z_B)$$

is the QGPV. Furthermore, since $u_0 = -\partial_y \eta_0, v_0 = \partial_x \eta_{0,}$, we have a single equation in η_0:

$$\begin{aligned} &\partial_t \Pi + J(\eta_{0,}\Pi) = curl\mathbf{E} \\ &\Pi = \gamma y + \nabla^2 \eta_0 - F(\eta_0 - z_B) \end{aligned} \qquad (2.1.5)$$

In the original dimensional variables, we denote the geostrophic component of the velocity as $(u_g, v_g) = U(u_0, v_0)$, and define the geostrophic streamfunction as

$$\partial_x \psi = v_g \equiv \frac{g}{f_0}\partial_x \eta, \quad \partial_y \psi = -u_g \equiv +\frac{g}{f_0}\partial_y \eta$$

This implies that the streamfunction is related to the surface elevation as

$$\psi \sim \frac{g}{f_0}\eta \qquad (2.1.6)$$

The QGPV is written in terms of ψ as

$$\boxed{\frac{D_g}{Dt} Q = curl\left(\frac{\mathbf{F}}{\rho}\right)} \qquad (2.1.7)$$

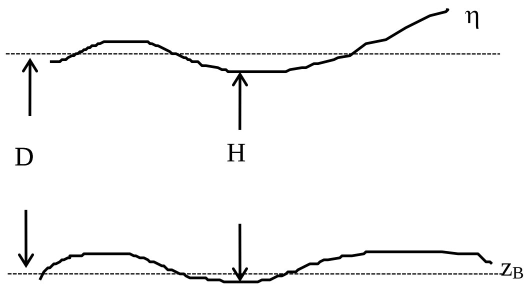
where

$$\frac{D_g}{Dt} Q = (\partial_t + u_g\partial_x + v_g\partial_y)Q = \partial_t Q + J(\psi, Q)$$

and the QGPV is

$$\begin{aligned} Q &= f_0 + \beta y + \xi_g - \frac{f_o}{D}(\eta - z_B) \\ &= f_0 + \beta y + \nabla^2 \psi - \frac{\psi}{L_D^2} + \frac{f_0}{D} z_B \end{aligned} \qquad (2.1.8)$$

Note 3 Derive QGPV from the SWPV.



With small surface elevation and bottom topography η/D, $z_B/D << 1$, we have

$$Q_{SW} = \frac{f+\xi}{H} = \frac{f+\xi}{D+\eta-z_B} = \frac{f+\xi}{D(1+\frac{\eta-z_B}{D})} \approx \frac{f+\xi}{D}(1-\frac{\eta-z_B}{D}) = \frac{1}{D}\left\{ f+\xi-\frac{(f+\xi)}{D}(\eta-z_B)\right\}$$

For large scale, $\xi / f = \varepsilon << 1$, we therefore have the relation between the SWPV and the QGPV as

$$Q_{SW} = \frac{1}{D}\{f+\xi-f(\eta-z_B)/D\} \approx \frac{1}{D}\{f+\xi_g-f(\eta-z_B)/D\}$$
$$= \frac{1}{D}\{f+\nabla^2\psi-f(\eta-z_B)/D\} \approx \frac{1}{D}\{f+\nabla^2\psi-\psi/L_D^2+f_o z_B/D\} = \frac{Q_{QG}}{D}$$

Note 4 Quasi-geostrophy and Geostrophy

On a f-plane, geostrophy has no divergence and vertical velocity at all. The quasi-geostrophy is geostrophy only at the leading order. Ageostrophic effect appears at order O(ε) as u_1, v_1, η_1. Indeed,

$$fv_1 - g\partial_x\eta_1 = -\varepsilon\left[(\partial_t + u\partial_x + v\partial_y)u_0 + \beta y v_0\right] \propto O(\varepsilon) << 1 \quad (but \neq 0)$$

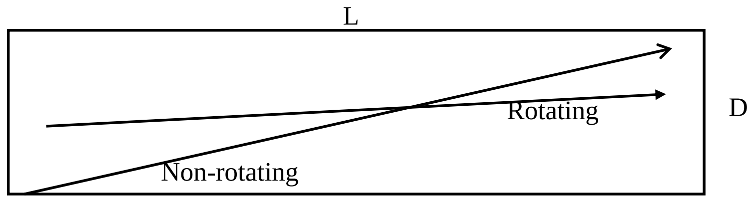
The divergence is also reduced because of the cancellation of $\partial_x u$ and $\partial_y v$,

$$\frac{\nabla\cdot\vec{v}}{\frac{U}{L}} \propto \frac{\frac{d\eta}{dt}\cdot\frac{1}{D}}{\frac{U}{L}} \propto \frac{\frac{N}{D}\frac{1}{L/U}}{\frac{U}{L}} \propto \frac{fUL}{gD} \propto \left(\frac{L}{L_D}\right)^2 \varepsilon \propto O(\varepsilon) << 1$$

Accordingly, the vertical velocity is also reduced

$$\frac{w}{u} \propto \frac{\frac{d\eta}{dt}}{u} \propto \frac{\frac{NU}{L}}{U} \propto \frac{\frac{f_0 UL}{g}}{L} \propto \frac{f_0^2}{gD}\cdot\frac{U}{f_0 L}\cdot\frac{D}{L}.L^2 \propto \left(\frac{L}{L_D}\right)^2 \cdot\varepsilon\cdot\frac{D}{L} \propto \varepsilon\frac{D}{L} << \frac{D}{L}$$

Thus, rotation suppresses divergence and vertical velocity (by O(ε)). Although small, however, the vertical motion is extremely important for the evolution of the system.



4. Energetics of a QG System

Before deriving the energy equation of the QG equation, we first notice the identity

$$-\psi\partial_t\partial_{xx}\psi = \partial_x(-\psi\partial_{xt}\psi) + \partial_t(\frac{1}{2}(\partial_x\psi)^2) \quad (2.1.9)$$

The energy equation of the QG model can be derived by multiplying -ψ on the QG equation as:

$$-\psi\partial_t Q - \psi J(\psi, Q) = 0.$$

With (2.1.9), we have

$$-\psi\partial_t Q = \partial_t\frac{1}{2}\left[(\psi_x)^2 + (\psi_y)^2\right] + \partial_x(-\psi\partial_{xt}\psi) + \partial_y(-\psi\partial_{yt}) + \partial_t(\frac{1}{2}\frac{\psi^2}{L_D^2}).$$

It is also straightforward that

$$\psi J(\psi, Q) = J(\psi^2/2, Q)$$

Notice

$$J(A, B) = \partial_x A\partial_y B - \partial_y A\partial B = \partial_x(A\partial_y B) - \partial_y(A\partial_x B)$$

and take x, y boundary conditions as rigid wall or periodic, we have

$$\partial_t\iint_A \left\{\frac{1}{2}\left[(\psi_x)^2 + (\psi_y)^2\right] + \frac{1}{2}(\frac{\psi^2}{L_D^2})\right\}dxdy = 0$$

This is the conservation of the total energy:

$$\partial_t\iint_A (KE + APE)dA = 0$$

where the QG KE and QG APE are

$$KE = \frac{1}{2}(\psi_x^2 + \psi_y^2), \quad APE = \frac{1}{2}\frac{\psi^2}{L_D^2}$$

The ratio between the KE and the APE is therefore

$$\frac{KE}{APE} = \frac{(\psi/L)^2}{(\psi/L_D)^2} \propto \left(\frac{L_D}{L}\right)^2$$

Therefore KE is comparable with the APE for synoptic scales, $L \propto L_D$, but becomes negligible relative to the APE for planetary scales $L >> L_D$.

5. Steady Geostrophic Flow

At steady state, QGPV becomes

$$J(\psi, Q) = 0$$

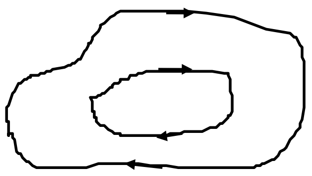
That is the flow is along Q-isoline, or

$$Q = Q(\psi).$$

Example 1:

On a f-plane, with a bottom topography $z_B(x,y)$,

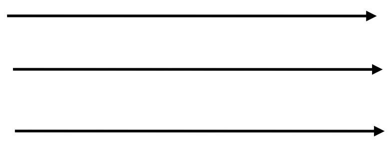
The streamline will be along the isobar.



Example 2:

On a beta-plane without bottom topography,

The steady flows are purely zonal.



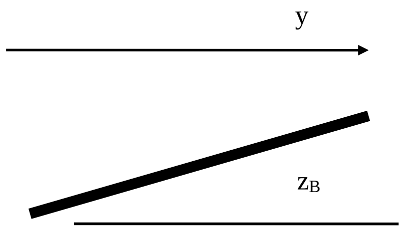
Sec.2.2: QG Rossby Waves

We now study small amplitude motions in the QG system. We will assume a mean flow without shear

$$\overline{\psi} = -\int^{y} U(y)dy = -U\ y$$

and a slope bottom topography varying only with latitude

$$z_B = \Lambda y + h_0\ .$$



The mean QGPV is

$$\overline{q} = f_0 + \beta y + \nabla^2\overline{\psi} - \frac{\overline{\psi}}{L_D^2} + \frac{f_0}{D}(\Lambda y + h_0)\,,$$

and its gradient is:

$$\overline{q}_x = 0,\quad \overline{q}_y = \beta + \frac{U}{L_D^2} + \frac{f_0\Lambda}{D} \equiv \overline{\beta},\quad \overline{q}_t = 0\ . \qquad (2.2.1)$$

The total streamfunction is separated into the mean and perturbation parts

$$\psi = \overline{\psi} + \psi'$$

with $\psi' << \overline{\psi}$. Accordingly, the PV is also separated into two parts

$$q = \overline{q} + q'$$

with $q' = \nabla^2\psi' - \frac{\psi'}{L_D^2}$. Notice that

$$J(\psi, q) \equiv J(\overline{\psi}, \overline{q}) + J(\overline{\psi}, q') + J(\psi', \overline{q}) + J(\psi', q') \approx J(\overline{\psi, q}) + J(\overline{\psi}, q') + J(\psi', \overline{q})\,,$$

in the absence of external source and sink, the linearized QGPV becomes

$$(\partial_t + U\partial_x)q' + \overline{\beta}\partial_x\psi' = 0\,, \qquad (2.2.2)$$

or

$$(\partial_t + U\partial_x)(\nabla^2\psi' - \frac{\psi'}{L_D^2}) + \overline{\beta}\partial_x\psi' = 0\,. \qquad (2.2.3)$$

This is a constant coefficient equation and therefore the solution can be assumed of the form $\psi' \sim \mathrm{Re}\{A\exp[i(kx + ly - \omega t)]\}$. Substitute this into (2.2.3), we have

$$\left[i(-\omega + Uk)(-K^2 - \frac{1}{L_D^2}) + ik\overline{\beta} \right] A = 0;$$

where $K^2 = k^2 + l^2$ is the total wave number. For nontrivial solutions, the amplitude remains nonzero A≠0. This gives the dispersion relationship for the Rossby wave as

$$\boxed{\omega = Uk - \frac{k\overline{\beta}}{K^2 + L_D^{-2}}} \tag{2.2.4}$$

or in phase speed

$$\boxed{c = \frac{\omega}{k} = U - \frac{\overline{\beta}}{K^2 + L_D^{-2}}} \tag{2.2.5}$$

The Rossby wave propagates only in one direction – westward, relative to the mean flow advection. This is in contrast to the Inertial-Gravity waves which propagate in all the directions. These I-G waves have been filtered out in the QG equation. Filtering out IG waves also implies an infinitely fast geostrophic adjustment time (or the IG wave propagates infinitely fast). As a result, the flow is always in geostrophic balance.

Waves of similar property can be found in the spherical coordinate, earlier by Haurwitz. The forced problem can be traced to the study of Laplace on tides about 150 years ago. However, it is Rossby who first realized that the beta-effect is the most important mechanism responsible for all the major features of these large scale waves.

1. Dispersion Relationship

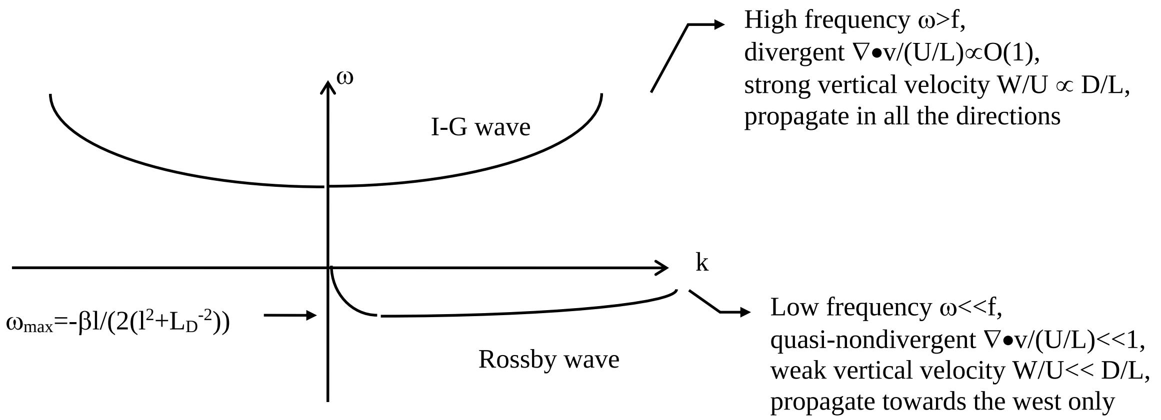
In the simplest case of $U=0$, $\Lambda=0$, we have $\overline{\beta} = \beta$. The dispersion relationship (2.2.4) becomes

$$\omega = -\frac{\beta k}{K^2 + L_D^{-2}} \quad \rightarrow \quad \begin{cases} 0, & when \quad k \rightarrow 0, \quad Longwave\ Limit \\ 0, & when \quad k \rightarrow \infty, \quad Shortwave\ Limit \end{cases}$$

The scale of the frequency satisfies

$$\frac{\omega}{f_0} \sim \frac{\beta}{L f_0} \min(L^2, L_D^2) \le \frac{\beta L}{f_0} \propto \frac{L}{a} << 1.$$

Therefore, the Rossby wave has a low (compared to 1/f) frequency, in contrast to the high frequency I-G waves. Indeed, on a f-plane, the Rossby wave is the zero-frequency geostrophic mode (see Sec.1.5), in other words, the geostrophic mode becomes Rossby wave when *f* is not a constant.



2. Barotropic Limit (Rigid lid approximation)

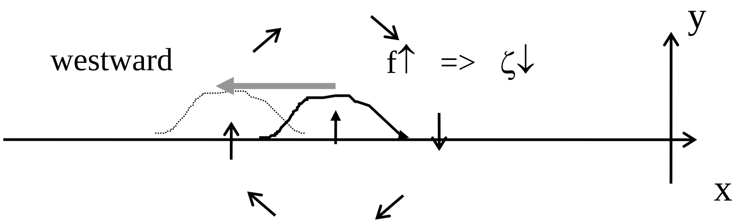
When the scale of the waves are much smaller than the deformation radius, $L<<L_D^2$, we have the dispersion relationship

$$\omega = Uk - \frac{\beta k}{K^2} \qquad (2.2.5)$$

This is the barotropic limit for Rossby waves, because barotropic Rossby waves have large deformation radius (thousands of kilometers). Now the flow is nearly nondivergent, because the free surface induced divergence, which is represented by $\eta \propto \psi/L_D^2$, is now negligible. The negligence of the free surface, however, does not mean the absence of surface pressure. One can imagine this case as a water with vanishing free surface elevation, but finite pressure gradient, or the rigid lid approximation. Since $L_D^2=gD/f^2$, the barotropic limit is easily realized in the limit of a deep water or strong interface gravity.

3. Mechanism of Rossby Wave Propagation and the "β-effect"

To consider the mechanism of Rossby wave propagation, we consider the simplest case of $U=0, \partial_y z_B = 0$, and $L<<L_D$, we have now $\overline{\beta} = \beta$ and the PV conservation becomes the conservation of absolute vorticity $\frac{d}{dt}(f+\zeta)=0$. A line of particles at latitude f_0 initially are at rest and therefore have the initial PV $q=f_0$. A northward perturbation of a particle will generate a negative relative vorticity $\zeta<0$, because of the PV conservation such that $f+\zeta=f_0$. The induced anticyclonic vorticity around this particle induces northward migration of particles to the west and therefore the perturbation appears to propagate westward.



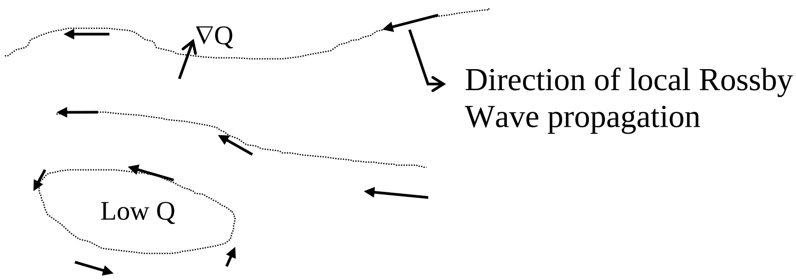
The discussion above also indicates that the restoring mechanism of the Rossby wave, in general, depends on the gradient of the background PV or the generalized beta, rather than the planetary vorticity alone. For example, bottom topography can generate an equivalent beta effect. Assuming $\beta = 0$, but $\partial_y z_B > 0$, we will have $\overline{\beta} = f_o \partial_y z_B / H > 0$, and the induced Rossby wave also propagates westward. Therefore, a northward shallowing topography has the same effect as the planetary beta, and therefore can be called the topographic beta.

What do we really mean by "westward" in the case of the generalized Rossby wave?

In general, the mean PV field $\overline{q}(x, y)$ can be of any shape. In the absence of advection, the generalized perturbation equation is

$$\partial_t q' + J(\psi', \overline{q}) = 0$$

The generalized Rossby wave will propagate "westward" if we assume the mean PV gradient ∇Q points towards the "north".



4. Non-Doppler-Shift Effect

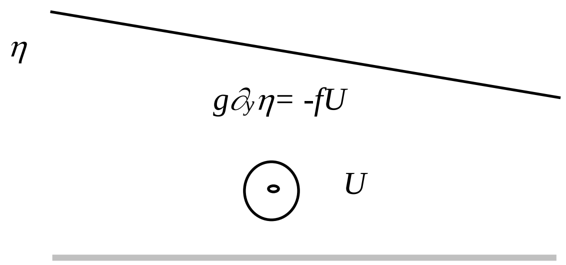
One interesting and peculiar feature of long Rossby wave is the so called Non-Doppler-shift effect. In the presence of advection, the Rossby wave speed is (2.2.5)

$$c = U - \frac{\overline{\beta}}{K^2 + L_D^{-2}}$$

where the first part is the advection effect or the Doppler shift effect, and the 2nd part is the generalized beta-effect. In the long wave limit, $k^2 << L_D^{-2}$, we have

$$c = U - \frac{\overline{\beta}}{K^2 + L_D^{-2}} = U - \frac{\overline{\beta}}{L_D^{-2}} = U - \frac{\beta + \frac{f_0 \Lambda}{D} + \frac{U}{L_D^{-2}}}{L_D^{-2}} = -\frac{\beta + \frac{f_0}{D}\Lambda}{L_D^{-2}}.$$

The wave speed is now independent of the mean flow *U*, the so called non-Doppler-shift effect! This apparent non-Doppler shift effect is due to the cancellation of the dual roles of the mean flow *U* that induces advection and mean PV gradient. Take a *U>0* as example. On the one hand, *U* advects the wave eastward; on the other hand, *U* is accompanied by a northward gradient of pressure, or a northward decrease of mean layer thickness. The latter enhances the planetary beta and therefore induces an additional westward propagation. This additional westward propagation cancels the eastward advection such that there is no net effect of the mean flow on the wave propagation.



It should be pointed out that the complete non-Doppler shift occurs here because both the flow and wave have the same vertical structure, the barotropic mode structure in the case of the shallow water system and the 1st baroclinic mode structure in the case of the 1.5-layer system. In the general case when the flow and the wave have different vertical structures, the complete non-Doppler shift effect does not exist anymore (see Chapter 5).

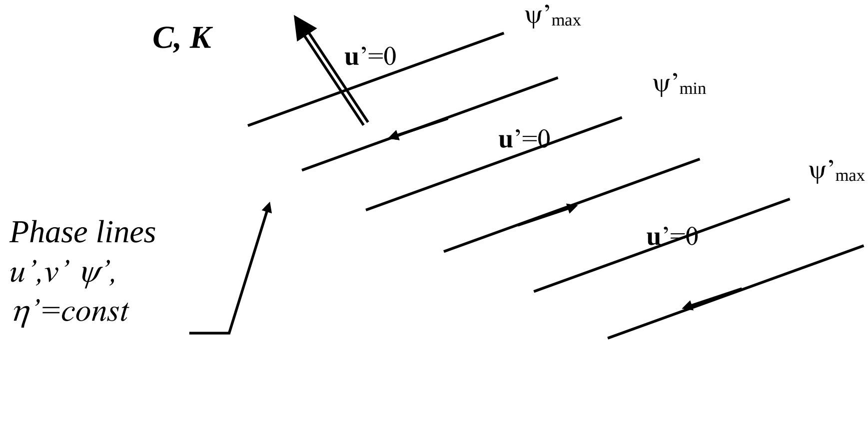
5. Wave structure

In general, the QG Rossby waves are transverse waves, because its velocity field is perpendicular to the direction of the wave vector (or phase propagation). With

$$u' = -\psi'_{y}, \quad v' = \psi'_{x},$$

we have $u' = -il\psi', v' = ik\psi'$, and therefore

$$\mathbf{u}'\cdot\mathbf{k} \sim (u', v')\cdot(k, l) = (-l, k)\cdot(k, l) = 0$$



Therefore, there is no self-advection

$$\because (u'\partial_x + v'\partial_y)q' \propto (uk + vl) \propto \mathbf{u}\cdot\mathbf{k} = 0$$

Thus, a plane wave is also an exact solution to the full nonlinear equation because now

$$\frac{D}{Dt}q = q'_t + U\partial_x q' + u'\partial_x Q + \mathbf{u}' \cdot \nabla q' = 0$$

(This is not true in the cases (i) on a sphere, (ii) with superimposition of plane waves and for iii) waves in shear and dissipation)

Sec. 2.3: Group Velocity and Energy Propagation of Rossby Waves

The most important reason that we study waves is that wave propagation is one of the two means by which fluid carries energy from one place to another (the other is advection). In the case of the geostrophic adjustment, it is the I-G wave that takes the ageostrophic part of energy away and therefore achieves the geostrophic balance. (see Sec. 1.6.)

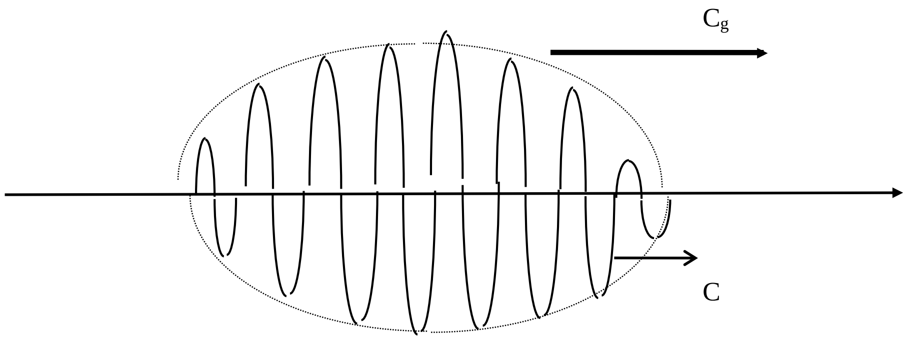
1. Group Velocity of Rossby Waves

Each single plane wave is valid only for a disturbance of infinite long wave patch. The phase speed for each plane wave only represents the speed of the phase. The energy of the wave, however, is represented by its amplitude, not its phase. In other words, the amplitude is represented by the envelop of the wave. The energy propagation speed therefore is the speed of the wave envelope, which could be different from the phase speed. The speed of energy propagation will be called the group velocity $\boldsymbol{C_g}$. The group velocity can be derived as

$$C_{gx} = \partial\omega/\partial k, \quad C_{gy} = \partial\omega/\partial l$$

In the case of the Rossby wave, take the case of $U=0$ as example, we have now

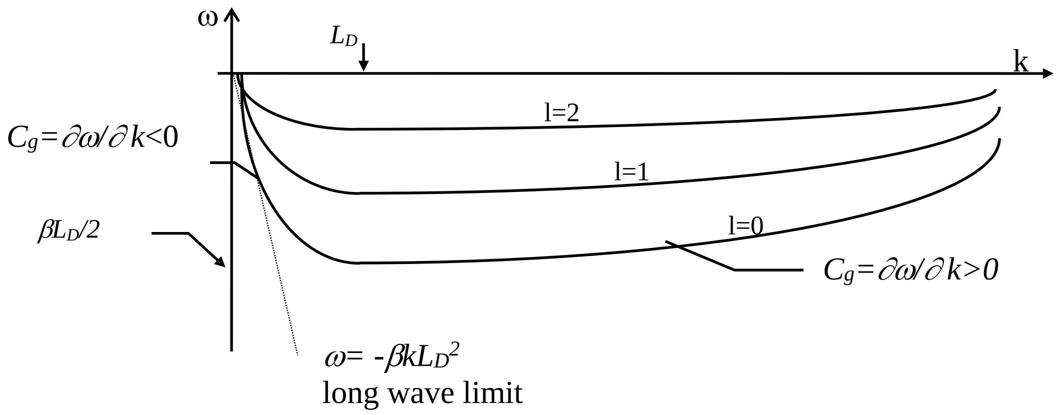
$$\omega = \frac{-\bar{\beta}k}{k^2 + l^2 + L_D^{-2}}$$



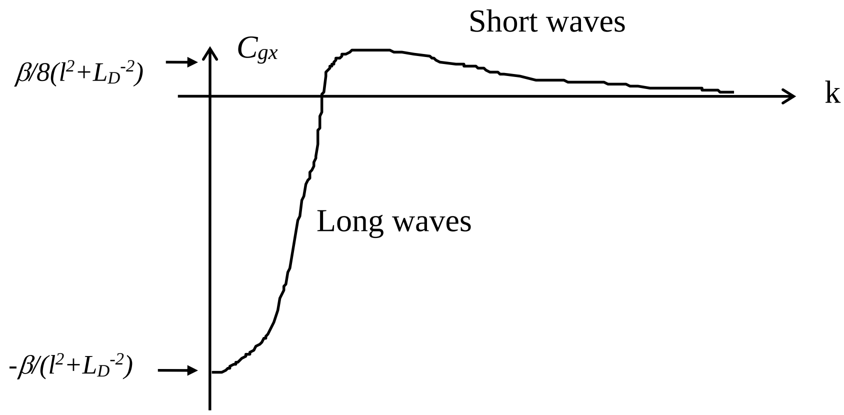
Propagation of a wave packet

For a given l, the maximum frequency occurs at $K_M^2 = l^2 + L_D^{-2}$ with $|\omega|_{\max} = \bar{\beta}/2K_m$.

The absolute maximum frequency (for all l) occurs at $l=0$ with $|\omega|_{\max} = \bar{\beta}L_D/2$, and $k_m = L_D^{-1}$. Therefore, the group velocity of the Rossby wave is westward for long waves, but eastward for short waves (although the phase velocity is always westward!). In addition, the maximum group velocity is 8 times faster towards the west than towards the east. This east/west asymmetry of group velocity has important implications to the general ocean circulation, which also has a strong east/west asymmetry (see Chapter 3).



Rossby wave dispersion relationship

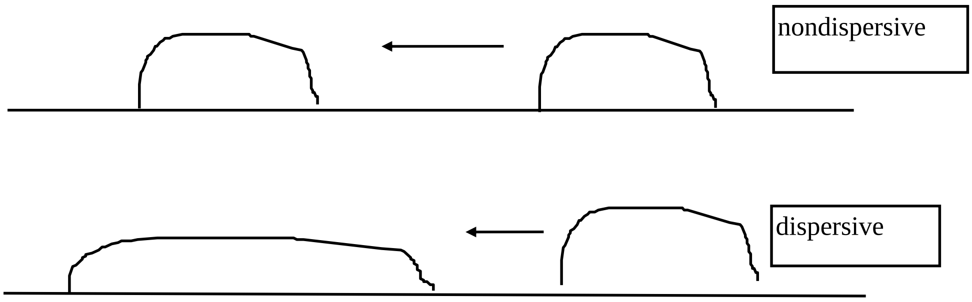


Rossby wave group velocity

At the long wave limit, $\omega = -\beta L_D^2 k$, we therefore have the group velocity the same as the phase velocity, $\frac{\partial\omega}{\partial k} = \beta L_D^2 = \frac{\omega}{k} = c$. The long Rossby waves are therefore nondispersive waves. The wave packet propagates without changing its shape, because all the single wave components propagate at the same speed. In general,

$$\frac{\partial\omega}{\partial k} = \frac{\partial(ck)}{\partial k} = c + k\frac{\partial c}{\partial k}.$$

Therefore, the wave is nondispersive only when $\frac{\partial c}{\partial k} = 0$. For a general dispersive wave, different wave component travels at different speed and therefore the initial wave packet will change shape and disperse. Notice that each single wave component extends into infinity, the shape of their summation is therefore virtually unpredictable after the initial time if the different component travels at different speed.



2. Energy Propagation Diagram

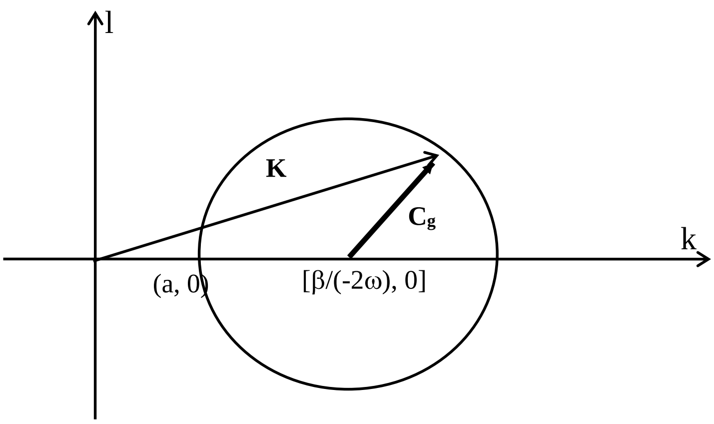
There is a convenient way to judge the propagation direction of Rossby waves. On the one hand, for a fixed frequency, the wave vectors falls on a circle in the (k,l) plane,

$$\omega = -\frac{\bar{\beta} k}{k^2 + l^2 + L_D^{-2}} \Rightarrow k^2 + l^2 + L_D^{-2} - \frac{\beta k}{(-\omega)} = 0$$

$$\left[k - \frac{\bar{\beta}}{2(-\omega)} \right]^2 + l^2 = \frac{\bar{\beta}^2}{4\omega^2} - L_D^{-2}$$

On the other hand, one can show the group velocity of a wave packet is parallel to the radius vector on this circle (pointing outward for k>0 and vise versa):

$$(c_{gx}, c_{gy}) = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}\right) = \frac{2\beta k}{(K^2 + L_D^{-2})^2} \left[k - \frac{\bar{\beta}}{2(-\omega)}, l \right]$$

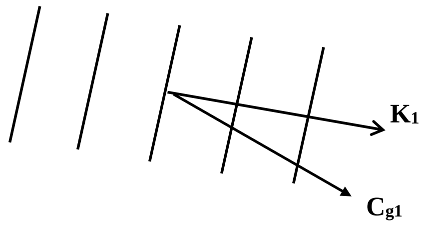


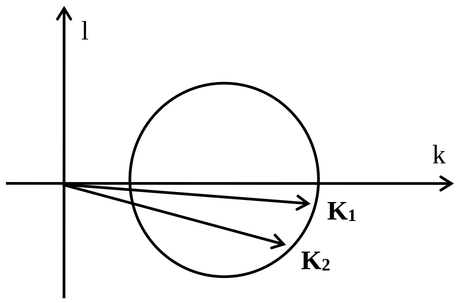
Rossby wave dispersion diagram

where $a = \beta/(-2\omega) - [(\beta/(-2\omega))^2 - L_D^{-2}]^{1/2}$

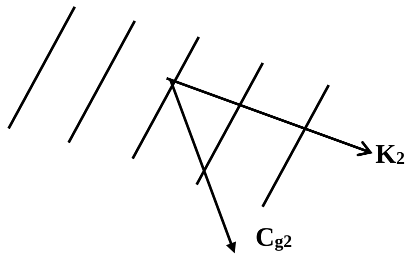
The dispersion diagram is very convenient for judging the direction of wave energy propagation based on the information of the wave phase.

Case 1:





Case 2:



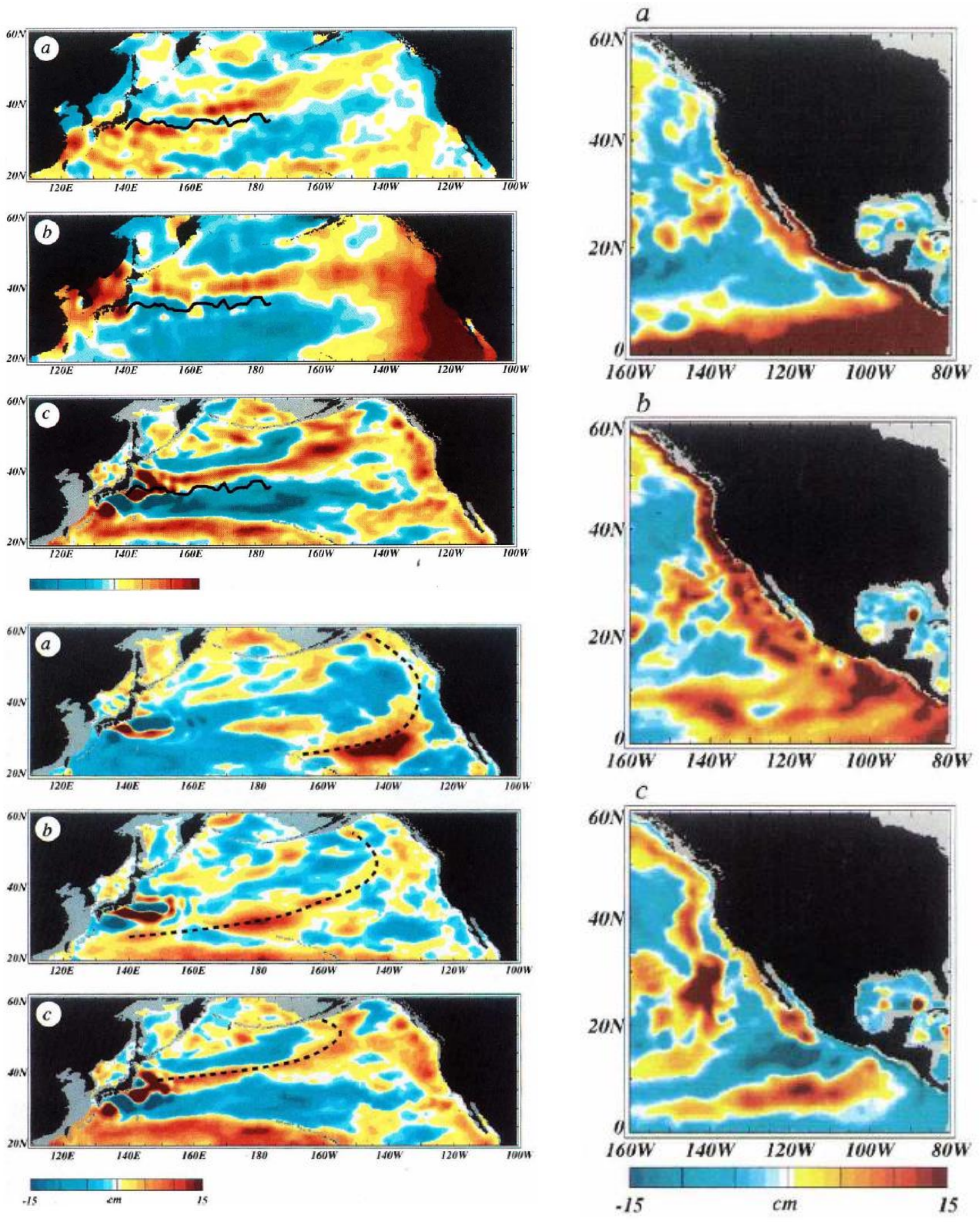


Fig.2.1 Oceanic Rossby wave propagation (Jacobs et al. 1994, Nature, Vol.370, P360-363)

Sea Surface Height Anomaly

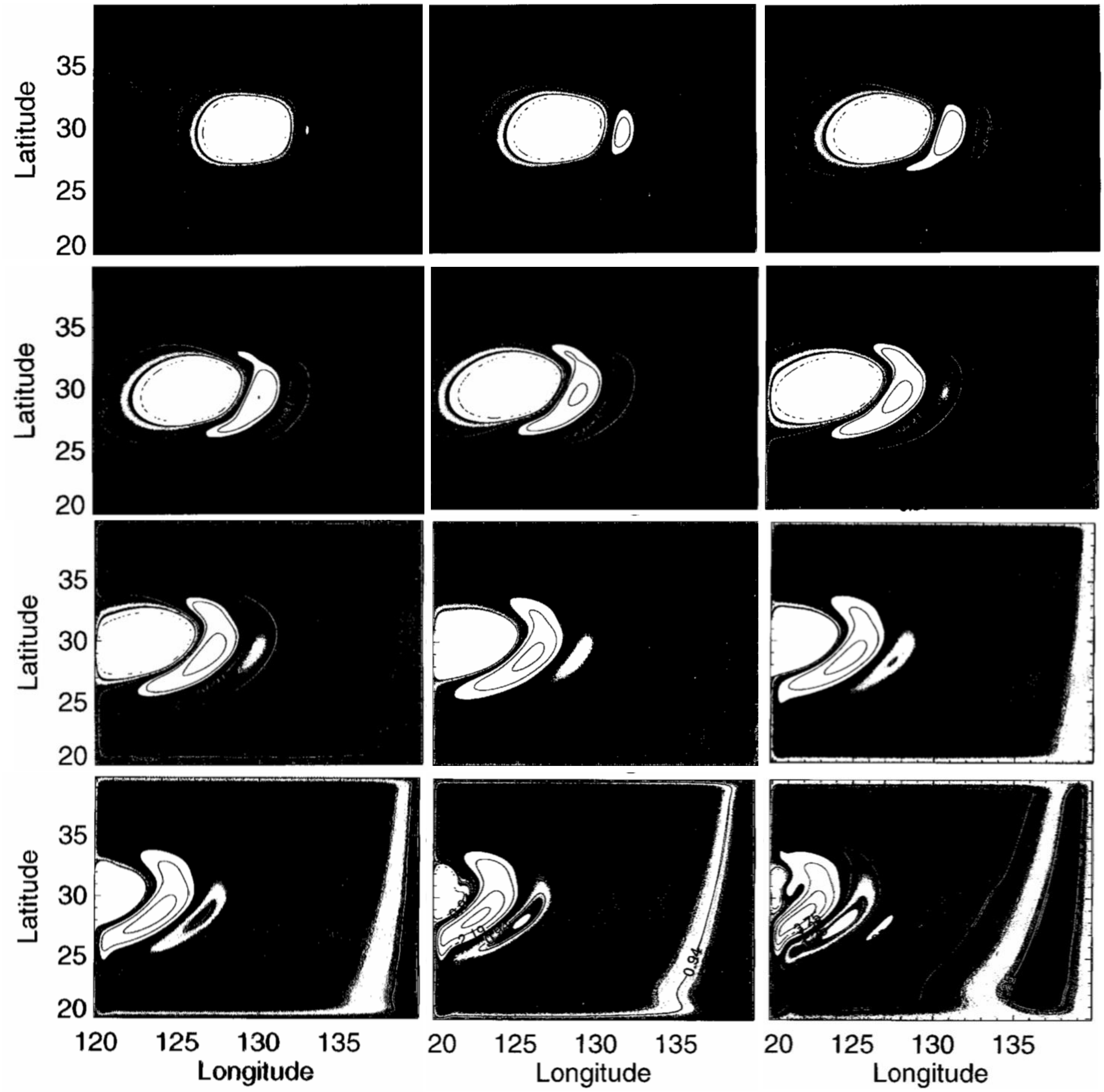


Fig.2.2 Propagation of a Rossby wave packet

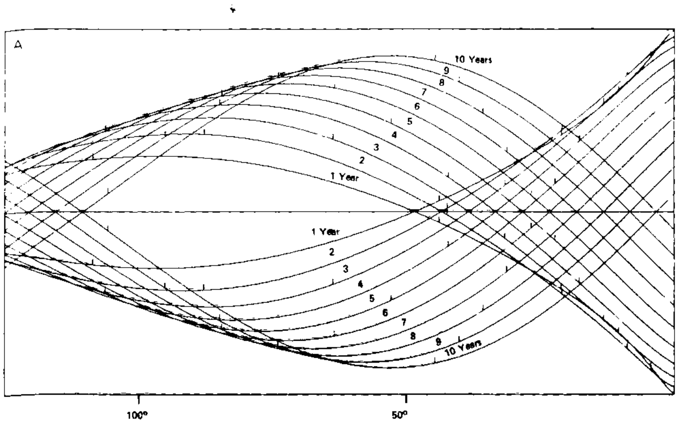


Fig. 8A. Time contours for semi-annual period waves generated at a meridional boundary. The tick marks indicate the latitude, at intervals of 5°, where the wave originated.

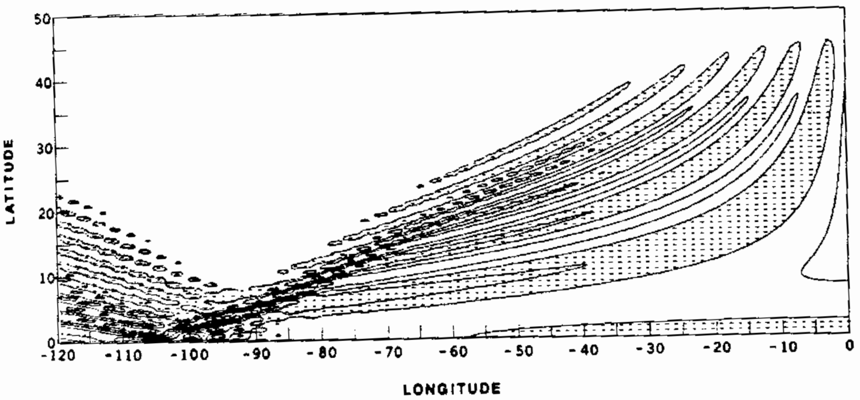


Fig. 9. Solutions to (2.1) calculated using equatorial modes. The value at $x = 0$ was used to determine the spectrum of the eigenfunction expansion up to a truncation limit of $m = 100$. The real part shows the solution at $t = 2n\pi$ years. The real part is contoured at $-0.1 \pm n$, $n = 1, 2, 3 \ldots$ avoiding the very small-amplitude, short-wave pattern in the shadow zone.

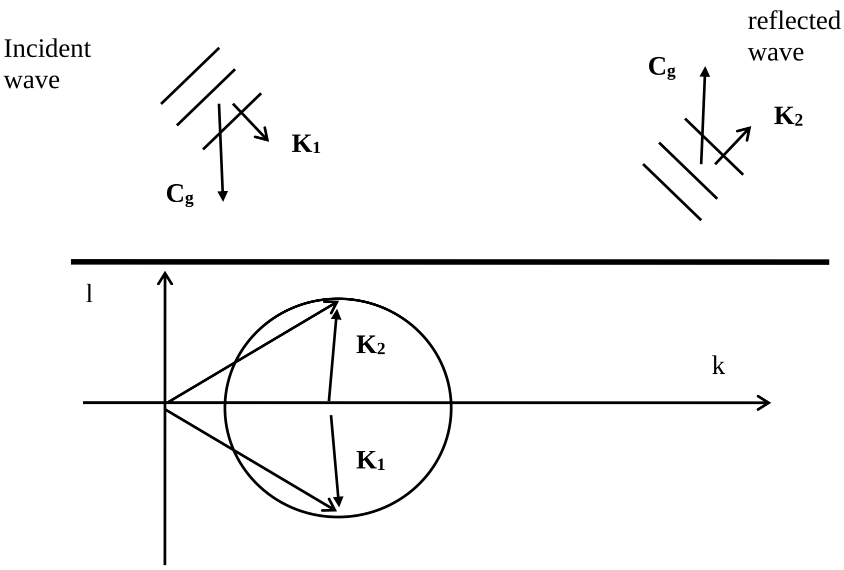
Fig.2.3 Rossby wave refraction (beta-dispersion)

Sec.2.4: Rossby Wave Reflection and Normal Mode

Here we study the reflection of a plane Rossby wave on a solid wall. Furthermore, we will study the wave field in a channel.

1. Reflection on y=const

First, we study the reflection on a zonal wall



Assuming an incident wave of the form:

$$\psi_1 = \mathrm{Re}\{A_1 e^{i(k_1 x + l_1 y - \omega_1 t)}\}$$

The energy of the incident wave propagates southward ($C_{gy}<0$, $l_1<0$) on the wall at $y=Y$.

The wave field ψ_1 itself does not satisfy the solid wall condition (no normal flow). Therefore, when it hits the boundary, it excites a reflected wave ψ_2 such that the total velocity field satisfy the boundary condition

$$\psi = \psi_1 + \psi_2 \,|_{y=Y} = 0\,. \qquad (2.4.1)$$

Since this condition is satisfied for all x and t, it is obvious that the frequency and along-shore wave number of the reflected wave are the same as the incident wave

$$k_2=k_1, \;\; \omega_2=\omega_1. \qquad (2.4.2)$$

Therefore, the boundary condition (2.4.1) reduces to

$$A_2 e^{il_2 Y} + A_1 e^{il_1 Y} = 0 \qquad (2.4.3)$$

Since both waves are free Rossby waves, they both satisfy the dispersion relationship, such that

$$-\frac{\beta k_1}{k_1^2 + l_1^2 + L_D^{-2}} = \omega_1 = \omega_2 = -\frac{\beta k_1}{k_1^2 + l_2^2 + L_D^{-2}} \qquad (2.4.4)$$

This gives

$$l_2 = \pm l_1 > 0$$

The final choice of l_2 depends on the energy radiation condition. The reflected wave has to propagate energy away from the wall, $C_{gy}>0$, opposite to the incident wave, to keep the total energy flux zero across the wall. Thus, we should have

$$l_2 = -l_1 > 0 \qquad (2.4.5)$$

and the amplitude of the reflected wave is derived from (2.4.3) as

$$A_2 = -A_1 e^{2il_1Y} \qquad (2.4.6)$$

The reflected wave is therefore

$$\psi_2 = -\mathrm{Re}\{A_1 e^{i(k_1x - l_1y - \omega_1 t)} e^{2il_1Y}\}$$

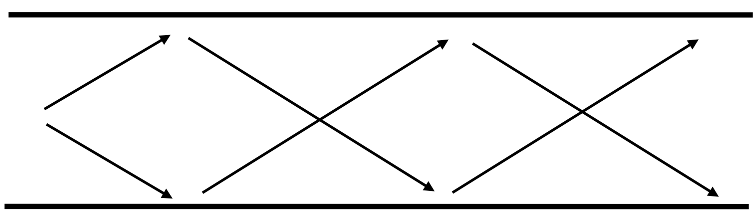
The total flow field is then

$$\psi = \mathrm{Re}\{A_1 e^{i(k_1x-\omega_1 t)}\left[e^{il_1y} - e^{-il_1y+2il_1Y}\right]\} = 2\,\mathrm{Re}\{A_1 e^{il_1Y} e^{i(k_1x-\omega_1 t)}\}\sin(l_1(y-Y)) \qquad (2.4.7)$$

Now, the boundary y=Y is a node point.

2. Zonal Channel

With two parallel walls, or in a zonal channel, the solution can be derived using the solution of a single wall reflection solution (2.4.7).



Taking the form of the solution (2.4.7), we have

$$\psi = A e^{i(kx-\omega t)} \sin(ly)$$

to satisfy the boundary condition at $Y=0$, and

$$\psi = A e^{ilLy} e^{i(kx-\omega t)} \sin[l(y - L_Y)]$$

to satisfy the boundary condition at $Y=L_y$. Since they have to be the same total wave field, which satisfy both boundary conditions simultaneously, we have

$$e^{ilLy} \sin[l(y-L_Y)] = \sin ly,$$

$$e^{ilLy}\left[e^{il(y-L_y)} - e^{-il(y-L_y)}\right] = e^{ily} - e^{-ily} \qquad \times e^{-ilL_y}$$

This leads to

$$e^{-il(y-L_y)} = e^{-il(y+L_y)}$$

$$e^{2ilL_y} = 1$$

i.e.

$$\cos(2lL_y) = 1, \quad \sin(2lL_y) = 0$$

Therefore, the meridional wave number are quantitized as

$$l = l_n = \frac{n\pi}{L_y}, \qquad n=1, 2, \tag{2.4.8}$$

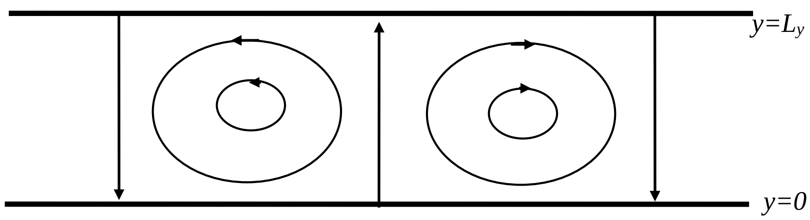
The total streamfunction is then

$$\psi = \mathrm{Re}\{Ae^{i(kx-\omega t)}\} \sin l_n y. \tag{2.4.9}$$

Compared with the half-plane solution (2.4.7), the solution (2.4.9) has two node points on the two walls. Furthermore, the cross-channel wave number is quantitized. Therefore, the wave forms normal modes in the y-direction. The normal modes are formed after many reflections on the two parallel walls. The key for the formation of the normal mode is that the wave energy has to be trapped between two boundaries.

<u>3. Periodic in X</u>

The channel condition is similar to a periodic condition as shown below.



Assuming the flow field is periodic in x with a length of L_x, that is $\psi(x) = \psi(x + L_x)$. The solution therefore satisfies

$$e^{ikx} = e^{ik(x+L_x)}$$

Therefore,

$$e^{iL_x k} = 1$$

or

$$\cos(kL_x) = 1, \sin(kL_x) = 0.$$

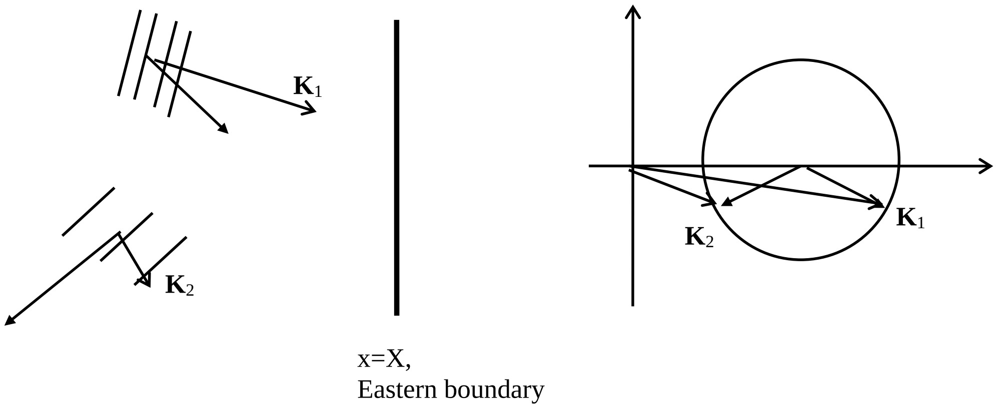
Therefore, the zonal wave number is also quantitized as

$$k_m = \frac{2m\pi}{L_x}$$

The periodicity condition guarantees the total energy conservation (energy flux out on one boundary is compensated by energy flux in from another boundary). This is similar to the case of a channel. Finally, in a zonal channel which is periodic in the y direction, the corresponding free mode (normal mode) has quantitized frequencies:

$$\omega = \omega_{m,n} = \omega(k_m, l_n), \qquad \text{m, n=1, 2, …}$$

4. Reflection on a *x* Boundary



An incident wave $\psi_1 = A_1 e^{i(k_1 x + l_1 y - \omega_1 t)}$ impinges on the eastern boundary *x=X*, the reflected wave is assumed of the form $\psi_2 = A_2 e^{i(k_2 x + l_2 y - \omega_2 t)}$. The solid wall reflection boundary condition requires $\psi=0$ on *x=X* for all the *y* and *t*. Thus, the meridional wave number and frequency of the reflected wave have to be the same as the incident wave

$$l_2 = l_1, \quad \omega_2 = \omega_1$$

This leads to the zonal wave number of the reflected wave as:

$$\frac{-\beta k_1}{k_1^2 + l_1^2 + L_D^{-2}} = \omega_1 = \omega_2 = -\frac{\beta k_2}{k_2^2 + l_2^2 + L_D^{-2}}$$

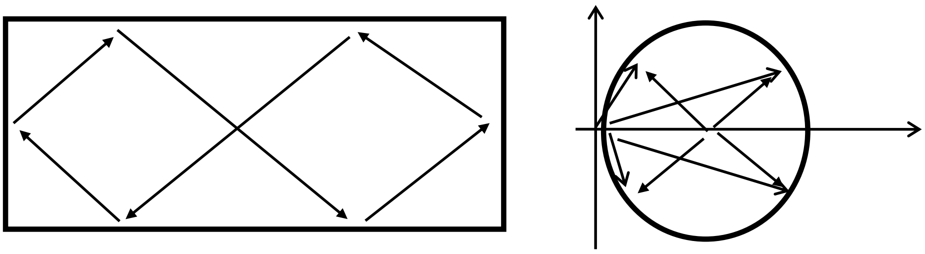
$$k_2^2 + \frac{\beta k_2}{\omega_1} + l_1^2 + L_D^{-2} = 0$$

$$k_2 = -\frac{\beta}{2\omega_1} \pm \sqrt{(\frac{\beta}{2\omega_1})^2 - (l_1{}^2 + L_D^{-2})}$$

where "+" is for the short wave and "-" for the long wave. Since the incident wave k_1 has the group velocity eastward (short wave), the reflected wave must have a group velocity westward and therefore is a long wave, which has $c_{gx} < 0$. This is necessary to oppose the eastward ($c_{gx} > 0$) incident wave energy flux.

5. Basin Mode

One can further discuss the Rossby wave modes within a basin. In principle, the presence of both the zonal and meridional channel walls quantitized the wave number in both the *x* and *y* directions



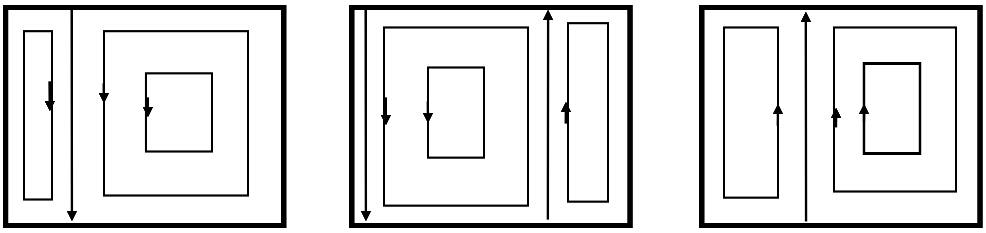
The net energy flux is zero in any direction. The reflection in both directions quantitized the wave number in both directions as.

$$\mathbf{K}_{m,n} = \left[\frac{\beta}{-2\omega} \pm \frac{m\pi}{L_x}, \pm \frac{n\pi}{L_y} \right]$$

One can find the basin mode as:

$$\psi \propto \cos\left\{ \frac{\beta x}{2\omega_{m,n}} + \omega_{m,n} t \right\} \sin \frac{m\pi x}{L_x} \sin \frac{n\pi y}{L_y}$$

The wave has a peculiar feature: it has westward phase propagation, but no net energy flux.



Sec.2.5: Forced Rossby Waves

In general, observed atmospheric and oceanic variability are caused either by external forcing or internal flow instability. The flow instability are caused by the shear of flows and the energy exchange with the mean flow. This will be discussed in chapter 6. Here we will focus on variability excited by external forcing such as wind stress and topography et al.. We will see that the free waves that we studied before are of critical importance in helping us understand these forced responses. In other words, the forced response can be understood in terms of free Rossby waves.

We consider two types of forced responses, all caused by a steady flow over mountains. The first type involves the flow over an isolated mountain while the second type over periodic mountain. The former concerns with the excitation of free waves and wave energy radiation into the far field, while the latter concerns with the resonance.

We first introduce the concept of stationary wave number. The general dispersion relationship of the Rossby wave (2.2.5) can be written as:

$$c = U - \frac{\bar{\beta}}{K^2 + L_D^{-2}} = \frac{U(K^2 - K_s^2)}{K^2 + L_D^{-2}} \tag{2.5.1}$$

where, in the absence of a mean topography,

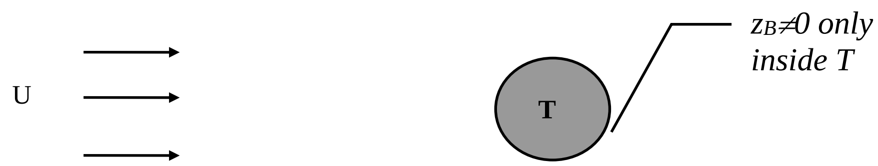
$$K_s^2 = \frac{\bar{\beta}}{U} - \frac{1}{L_D^2} = \frac{\beta}{U} \tag{2.5.2}$$

is the stationary wave number. With a typical wind of *U=10m/s*, the corresponding wavelength is about 5000km A general Rossby wave propagates in different directions according to its wave length relative to that of the stationary wave.

long wave,	$K^2 < K_s^2 \Rightarrow c < 0$,	westward propagation,
stationary wave	$K^2 = K_s^2 \Rightarrow c = 0$,	stationary
short wave	$K^2 > K_s^2 \Rightarrow c > 0$,	eastward propagation.

1. Flow over isolated mountain

Consider an infinite beta-plane, a uniform mean current *U* passes over an isolated mountain *T*. It is straightforward that the flow field locally near the mountain has to be distorted. The most interesting question here is if the mountain can also generate remote responses away from the mountain.



Except inside the isolated *T*, $z_B=0$, the response, if available, is simply free Rossby waves of the form:

$$\psi = \mathrm{Re}(\hat{\psi} e^{i(kx+ly)-ikct}) \tag{2.5.3}$$

with $c = U(K^2 - K_s^2)/(K^2 + L_D^{-2})$

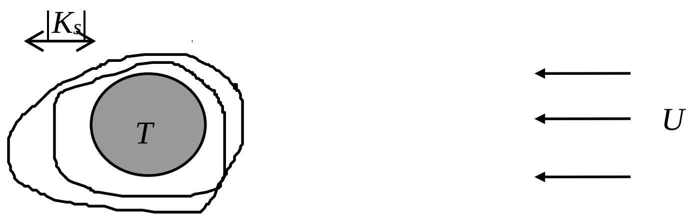
Since $c=0$ (stationary) for the fixed mountain, we have the wave length of the forced response as

$$K^2 = k^2 + l^2 = K_s^2(U)\,. \tag{2.5.4}$$

The remote response depends on the stationary wave number or in turn the mean flow conditions.

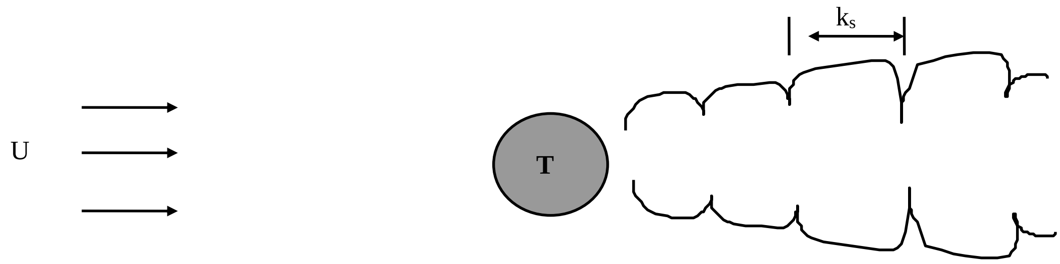
(i) Easterly wind, $U<0$:

Under an easterly wind, we have from (2.5.2) $K_s^2 < 0,$. The forced response (2.5.3) has an imaginary wave number. Since the wave energy originates from the mountain, the forced response can't be infinitely large away from the mountain. The only possibility is that the forced response decays away from the mountain. Therefore, the response is an evanescent solution with only localized responses. The flow ψ decays with distance from *T*. In other words, under the easterly wind, no free waves exist to match the stationary wave number. So there is no wave energy radiating away from the mountain. The response in the far field is weak.



(ii) $U>0$

With a mean westerly wind, $K_s^2 > 0$ according to (2.5.2). The forced response has a real wave number, corresponding to propagating solutions. This free Rossby wave radiates energy into the far field and produces strong response there. The conclusion that only westerly wind can generate downstream response can also be understood from the handwaving argument in terms of the conservation of potential vorticity (see Holton, Figs.4.9,4.10).



The example above demonstrates a general principle. For an isolated disturbance $e^{i(kx-\sigma t)}$, if the forcing frequency σ can excite free wave (with real wave numbers), the disturbance can be radiated away to the far field. Otherwise, the response is trapped near the wave maker.

The direction of the wave energy propagation can be further studied in the case of $K_s^2>0$.

From the dispersion relation (2.5.1), $c = U\dfrac{K^2 - K_s^2}{K^2 + L_D^2}$, we have the group velocity for stationary waves ($c=0$) as:

$$c_{gx} = \partial_k \omega = c + k\frac{\partial c}{\partial k} = k\frac{\partial c}{\partial k} = \frac{2k^2\bar{\beta}}{(K^2 + L_D^{-2})^2} > 0$$

Thus, wave energy always radiates to the east of the mountain.

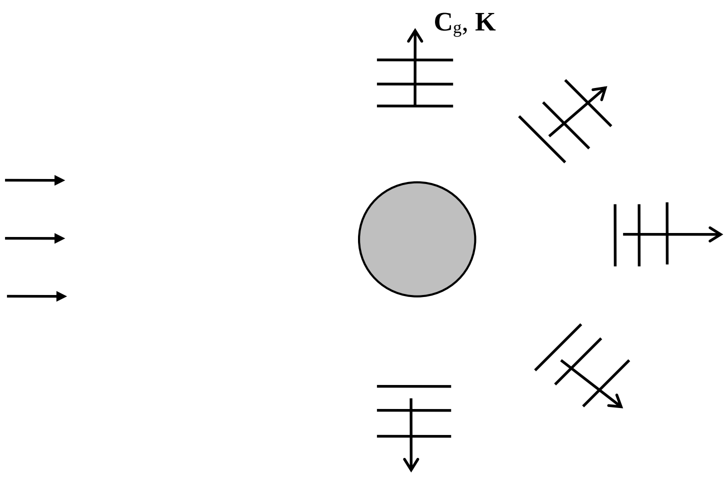
Furthermore, we have

$$c_{gy} = \partial_l \omega = k\frac{\partial c}{\partial l} = \frac{2kl\bar{\beta}}{(K^2 + L_D^{-2})^2},$$

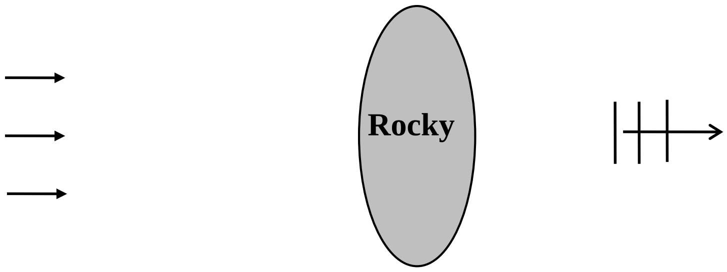
Therefore, the group velocity can be written as

$$\mathbf{c}_g = \frac{2k\bar{\beta}}{(K^2 + L_D^{-2})^2}\mathbf{k}$$

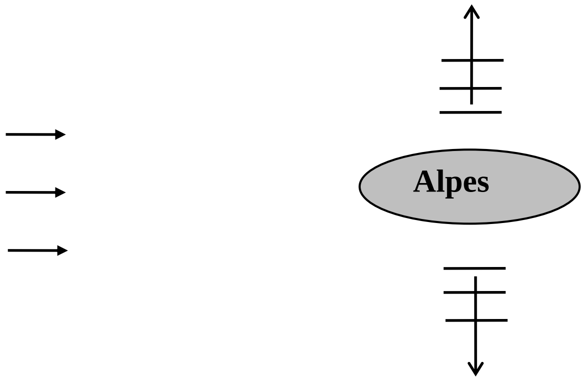
(This is obvious if one notices in (2.5.1) that $c=c(K^2)$). That is: the group velocity of the stationary Rossby wave is in the same direction as the wave vector $\boldsymbol{K}$!



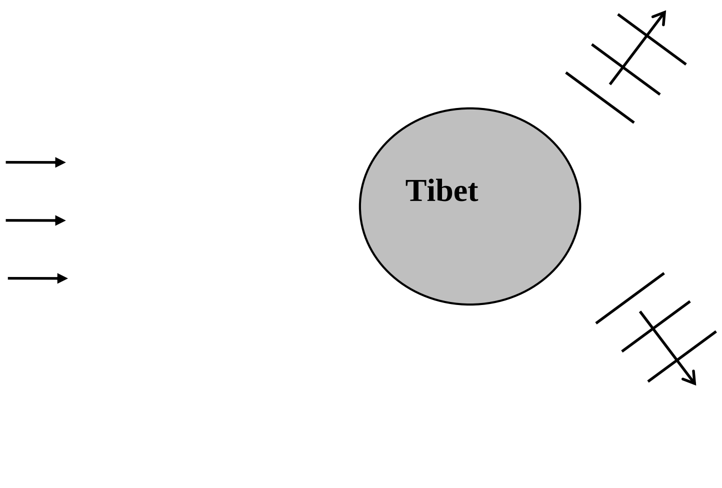
The dominant direction of the propagation depends on k/l, which will be determined by the shape of the mountain, or more precisely its dominant projection on $e^{i(kx+ly)}$. For a mountain like the Rocky Mountain, the dominant projection has $k>>l$, because of its dominant north/south elongated shape. Therefore, the dominant stationary wave response is eastward downstream.



The Alpes is the opposite, with $l>>k$, because of its dominant east/west elongated shape. The dominant response therefore is north/south.

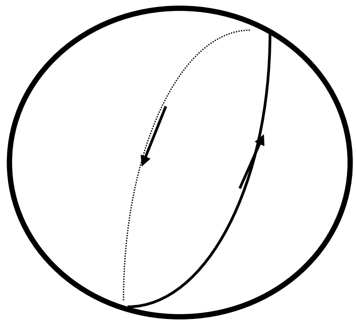


The Tibet is more rounded with k comparable with l. The resulted response therefore tends to radiate in the northeast and southeast directions.

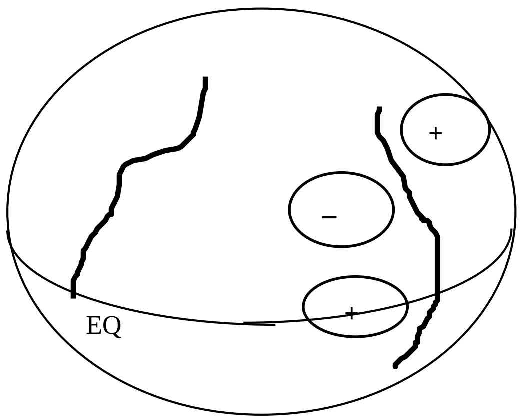


Note 1. Stationary Rossby wave propagation on a sphere and Teleconnection:

The propagation of stationary Rossby waves plays a critical role in climate study. At the climate time scale, we can treat the Rossby waves virtually as stationary. Its propagation can relate the climate in one part of the world to that in the other part. This is called atmospheric "teleconnection". The atmospheric teleconnection becomes particular complex on a sphere. For a simple mean flow, one can show that the planetary wave propagates along a great circle (Hoskins and Karoly, 1981).



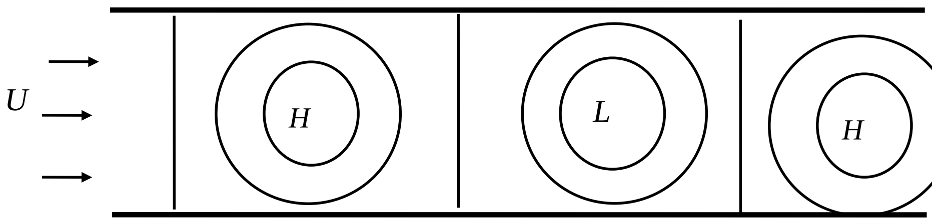
An important observational evidence of the atmospheric teleconnection is the PNA (Pacific/North America Pattern). This atmospheric teleconnection pattern enables the eastern equatorial Pacific SST anomaly, as occurred during the El Nino years, to affect the climate in the North America. Here, the forcing is a local thermal anomaly.



Several things need to be kept in mind. First, cross-equator propagation is usually prohibited by the dominant easterly wind there. Second, if dissipation is strong enough, the waves will be damped heavily before it propagates far away. Third, similar wave radiation can be found in the ocean, such as the Gulf Stream eddy radiation.

2. Flow over sinusoidal topography

Now, we consider a mean zonal flow over a periodic mountain within zonal channel.



The bottom topography can be represented as

$$z_B(x,y) = \Delta \sin ly \cos kx, \qquad l = \frac{\pi}{L},$$

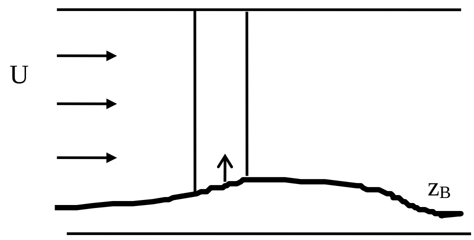
where the amplitude of the mountain has been assumed small relative to the total depth

$$\frac{\Delta}{D} << 1$$

such that the dominant flow climbs over the topography (instead of circling the topography) and represents a weak linear response. Thus, the basic state has no topography $z_B = 0$ and the mean PV field is simply

$$\bar{q} = f_0 + \beta y + \frac{Uy}{L_D^2}$$

$$\bar{q}_y = \beta + \frac{U}{L_D^2} = \bar{\beta}$$



The topography affects the perturbation PV

$$q' = (\nabla^2 - \frac{1}{L_D^2})\psi + \frac{f_0}{D} z_B .$$

Assuming the dissipation is a Rayleigh damping,

$$\frac{\mathbf{F}}{\rho} = -\lambda(u,v), \quad or \quad curl\frac{\mathbf{F}}{\rho} = -\lambda\nabla^2\psi$$

The linearized PV equation for the perturbation is

$$(\partial_t + U\partial_x)q' + v'\overline{\beta} = -\lambda\nabla^2\psi$$

or in terms of the streamfunction as

$$(\partial_t + U\partial_x)(\nabla^2 - \frac{1}{L_D^2})\psi + \lambda\nabla^2\psi + \partial_x\psi\overline{\beta} = -\frac{f_0 U}{D}\partial_x z_B \qquad (2.4.6)$$

For convenience, we write the topography as

$$z_B = \mathrm{Re}\{\Delta e^{ikx} \sin ly\},$$

the forced response will take the form of

$$\psi = \mathrm{Re}\{\hat{\psi} e^{ikx} \sin ly\}.$$

Substitute these into the equation, we have

$$\left\{-ikU(K^2+L_D^{-2})-\lambda K^2+ik\overline{\beta}\right\}\hat{\psi}=-ik\frac{f_0U}{D}\Delta .$$

This gives the amplitude of the forced response as

$$\hat{\psi}=\frac{f_0U}{D}\Delta\left\{U(K^2+L_D^{-2})-\overline{\beta}-\frac{i\lambda K^2}{k}\right\}^{-1}$$

$$=\frac{f_0U}{D}\Delta\left\{U(K^2-K_S^2)-\frac{i\lambda K^2}{k}\right\}^{-1}$$

The corresponding surface elevation is therefore:

$$\hat{\eta}=\frac{f_0}{g}\hat{\psi}=\frac{\Delta}{L_D^2}\left\{(K^2-K_S^2)+\frac{i\lambda K^2}{Uk}\right\}^{-1} \quad (2.4.7)$$

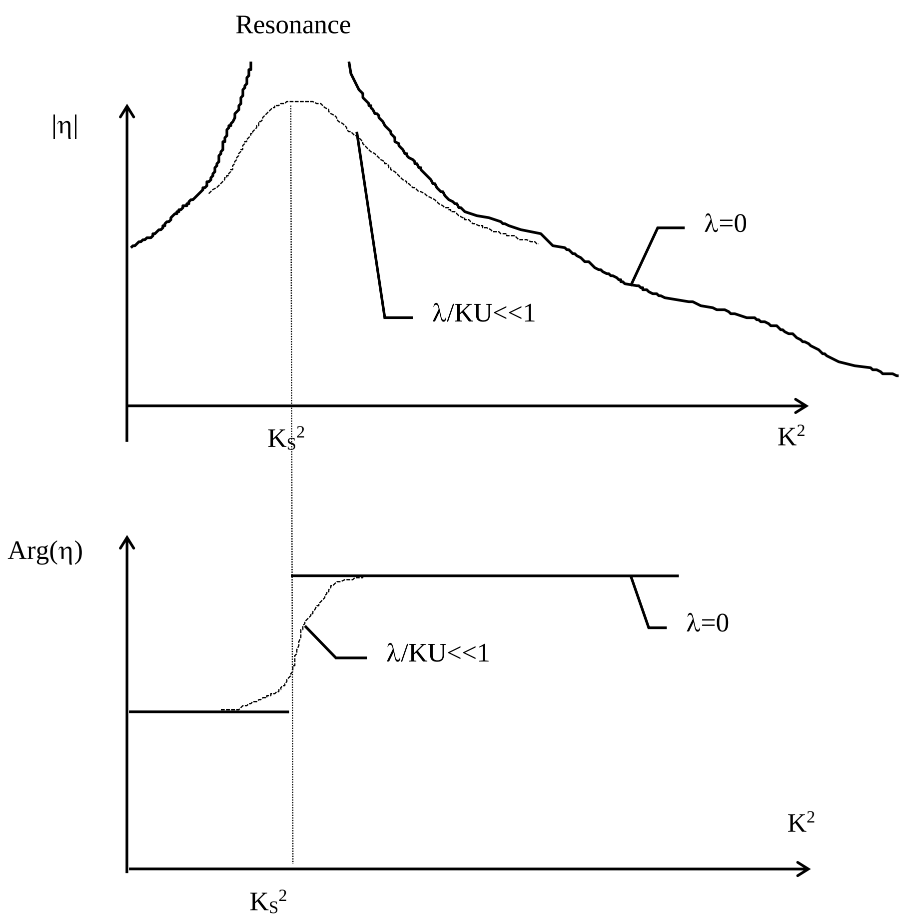
Without friction, $\lambda = 0$, we have the amplitude as

$$\hat{\eta}=\frac{\Delta}{L_D^2(K^2-K_S^2)} \quad (2.4.8)$$

When the forcing wave number is the same as the stationary wave number $K=K_S$, the amplitude of the response is infinite and the phase has an abrupt change across K_S. This is the resonance response. In general, with dissipation, the response amplitude is finite. The maximum amplitude depends on the ratio of the dissipation and advective time scales. The larger the dissipation time scale, the larger the response amplitude. In the case of weak dissipation,

$$\frac{\lambda}{KU}\propto\frac{1/\tau_{Diss..}}{1/\tau_{Adv.}}=\frac{\tau_{Adv.}}{\tau_{Diss.}}<<1,$$

the amplitude is finite and the phase still shifts abruptly across the resonant wave number.



In contrast to the isolated mountain case, now if the forcing structure fits the free wave, it generates resonant responses. This is also a general principle when the forcing is applied onto every point of the flow field. In the former case, the forcing is applied to an isolated region and the response, when free wave is excited, appears as remote responses in the far field, but with finite amplitude. In the latter case, the forcing is applied everywhere on the fluid, and the response, when free wave is excited, exhibits an amplified amplitude. In both cases, the understanding of the free wave is of critical importance for us to predict the response for a given forcing.

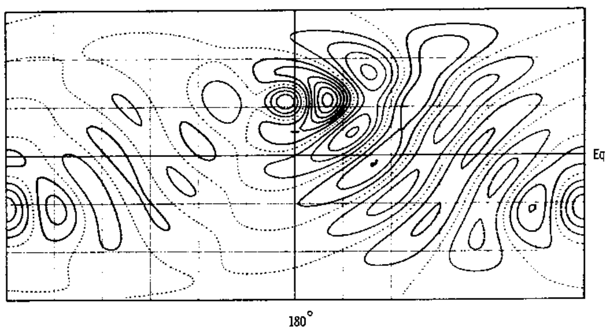


Fig. 11.6. Planetary wave propagation on a sphere, as found in a numerical experiment of Grose and Hoskins (1979). Contours are of perturbation vorticity, and disturbances to a superrotation zonal flow (i.e., an eastward flow with uniform angular velocity about the earth's axis) are produced by a circular mountain centered at 30°N and 180° longitude, and with radius equal to 22.5° of latitude. Waves travel backward and forward across the equator along ray paths that are curved because of variation in the Coriolis parameter f with latitude. The equatorial trapping effect is evident. The amplitude of the wave decays with distance because of dissipative effects included in the model. [From Grose and Hoskins (1979, Fig. 3a).]

Fig.2.4 Stationary Rossby wave response and atmospheric teleconnection

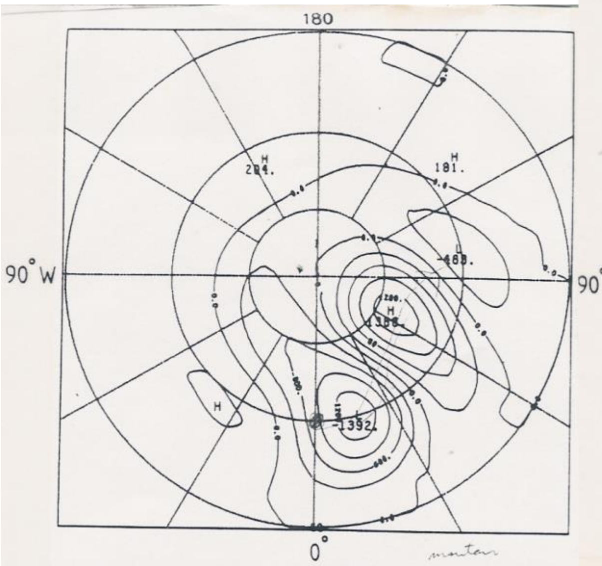


FIG. 1. Geopotential response to a localized vorticity forcing centered on the Greenwich meridian at latitude 30°N. Solution of (6.2) with the forcing (6.5). Amplitude is arbitrary.

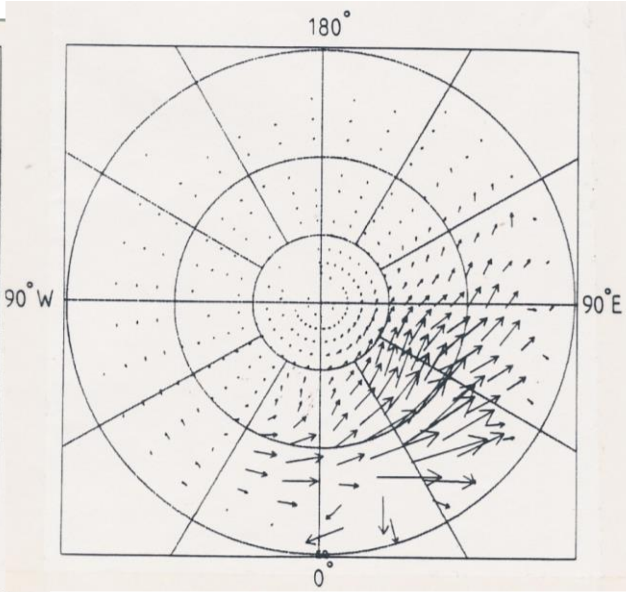


Fig.2.5 Stationary Rossby wave on the sphere and E-P flux

*Section 2.6: Non-plane Waves

We have so far been studying plane waves. These waves only exist on an infinite domain and in homogenous medium. In a more general domain and medium, a wave can be represented as the summation of various plane waves:

$$\psi = \mathrm{Re}\{\int \hat{\psi}(k,l)e^{i(kx+ly-\omega t)}dkdl\} \tag{2.6.1}$$

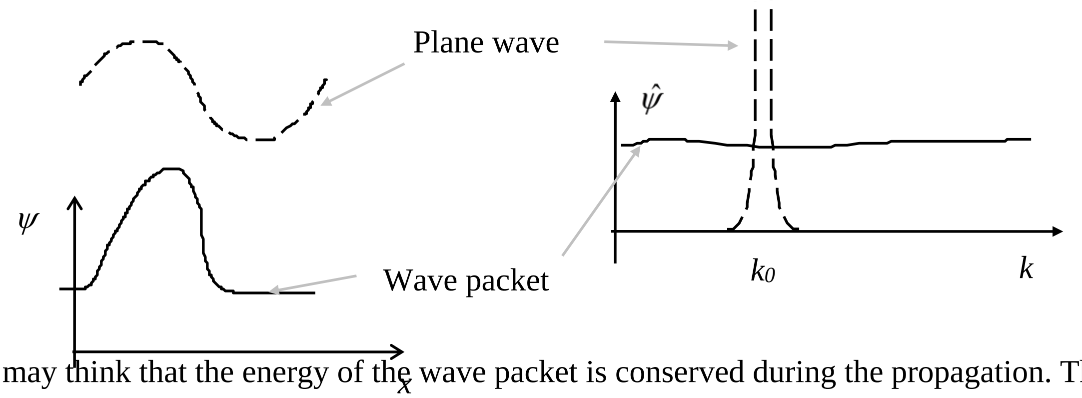
The plane-wave case is the special case with the spectrum $\hat{\psi}$ being a δ-function:

$$\hat{\psi} = \delta(k-k_0, l-l_0), \tag{2.6.2}$$

the general wave (2.6.1) reduces to a single wave

$$\psi = \mathrm{Re}\{e^{i(k_0x+l_0y-\omega t)}\} \tag{2.6.3}$$

Such plane waves are rarely seen in reality because it requires homogeneous medium and an infinite domain. For more general cases, the wave group is a wave packet, which consists of waves at many wavelengths and we are most concerned with the propagation of the wave packet in an inhomogeneous medium.



You may think that the energy of the wave packet is conserved during the propagation. This is not true if the mean flow has shear, so there is wave-mean flow interaction. A more general conservation quantity, however, is the wave activity.

1. Conservation of Wave Activity

Consider a basic state of $\Psi(y)$, the mean flow is $U = -\frac{d\Psi}{dy} = U(y), \quad V = 0$. The background QGPV is:

$$Q(y) = f_0 + \beta y + \frac{d^2\Psi}{d^2y} - \frac{\Psi}{L_D^2}$$

For a small disturbance ψ', we have the streamfunction

$$\psi = \Psi + \psi'(x,y,t)$$

where $\psi' << \Psi$ and the flow

$$u = U + u'$$

where $\left|\frac{u'}{U}\right|$ <<1 (more precisely $\left|\frac{u'}{U-c}\right|$ <<1).

For later convenience, we first rewrite the northward QGPV flux $\overline{v'q'}$ in the convergence form.

$$q' = \nabla^2\psi' - \frac{\psi'}{L_D^2},$$

$$v'q' = \partial_x\psi'\left\{\partial_{xx}\psi' + \partial_{yy}\psi' - \frac{\psi'}{L_D^2}\right\}.$$

Using the identities

$$\psi'_x\,\psi'_{xx} = \partial_x\frac{1}{2}(\psi'_x)^2,$$

$$\psi'_x\,\psi'_{yy} = \partial_y(\psi'_x\,\psi'_y) - \psi'_y\,\psi'_{xy} = \partial_y(\psi'_x\,\psi'_y) - \partial_x\left[\frac{1}{2}(\psi'_y)^2\right], \tag{2.6.4}$$

$$\psi'_x\,\psi' = \partial_x\left[\frac{1}{2}(\psi')^2\right],$$

we can write the PV flux as

$$v'q' = \partial_x\left\{\frac{1}{2}(\psi'_x)^2 - \frac{1}{2}(\psi'_y)^2 - \frac{\psi'^2}{2L_D^2}\right\} + \partial_y(\psi'_x\,\psi'_y) \tag{2.6.5}$$

Defining the E-P flux vector $\boldsymbol{F}$ as

$$\mathbf{F} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}D\rho_0[v'^2 - u'^2 - \frac{\psi'^2}{L_D^2}] \\ -D\rho_0 u'v' \end{Bmatrix} \tag{2.6.6}$$

the PV flux can be written as the divergence of the E-P flux

$$D\rho_0 v'q' = \nabla \bullet \mathbf{F} \tag{2.6.7}$$

Now, we return to the linearized QGPV equation.

$$(\partial_t + U\partial_x)q' + v'Q_y = G$$

Multiplied by q', we have

$$(\partial_t + U\partial_x)\frac{q'^2}{2} + v'q'Q_y = q'G$$

Again multiplied by $\rho_0 D / Q_y$ (which is a function of y), we have

$$(\partial_t + U\partial_x)\frac{\rho_0 D}{2}\frac{q'^2}{Q_y} + D\rho_0 v'q' = \frac{D\rho_0 q'G}{Q_y} = S_A$$

Therefore, we have:

$$\boxed{(\partial_t + \mathbf{U} \bullet \nabla)A + \nabla \bullet \mathbf{F} = S_A} \quad (2.6.8)$$

where

$$A = \frac{1}{2} D\rho_0 \frac{q'^2}{Q_y} \quad (2.6.9)$$

is the wave activity density per unit area. Since $\nabla \bullet \boldsymbol{U} = 0$ for QG flows, we can rewrite (2.6.8) as:

$$\boxed{\partial_t A + \nabla \bullet (\boldsymbol{U}A + \boldsymbol{F}) = S_A} \quad (2.6.10)$$

This is the wave activity equation. It is important to realized that the equation is valid as long as $\psi' << \Psi$ and there is no restriction on ψ' being a plane wave at all. This equation is also generalized Eliassen-Palm equation. The flux $\boldsymbol{U}A + \boldsymbol{F}$ is the wave activity flux, the first part is due to advection by the mean flow, while the second part due to wave radiation relative to the mean flow.

Since A and $\boldsymbol{F}$ are of quadratic form of the perturbation, they will have a higher harmonic/frequency component. In practice, A and $\boldsymbol{F}$ are made more manageable by some kind of average (depending on the problem), such as the *x*-average, *t*-average or average over a wave length. (This kind of average makes no sense to a variable that is of linear on the perturbation, because it is always zero). Here, we take the zonal average, as in most cases.

$$\overline{(p)} = \frac{1}{2L} \int_{-L}^{L} P(x, y, t) dx = \overline{P}(y, t).$$

Assuming either the disturbance vanishes at infinity, or the system is periodic in *x* such that *P(-L)=P(L)*, we have

$$\overline{\left(\frac{\partial P}{\partial x}\right)} = \frac{1}{2L}\left[P(L) - P(-L)\right] = 0$$

Averaging the wave activity equation (2.6.10), we have

$$\boxed{\partial_t \overline{A} + \partial_y (\overline{F}_y) = \overline{S}_A} \quad (2.6.11)$$

This is the shallow water version of the Eliassen-Palm equation, where the wave activity is

$$\overline{A} = \frac{1}{2} \rho D_0 \frac{\overline{q'^2}}{Q_y} \quad (2.6.12)$$

and the E-P flux is

$$\overline{F}_y = -\rho_0 D(\overline{u'v'}). \quad (2.6.13)$$

If further there is no source and sink, $S_A=0$, we have the conservation of wave activity as

$$\partial_t \overline{A} + \partial_y (\overline{F}_y) = 0 \quad (2.6.14)$$

or $\partial_t \overline{A} = -\partial_y(\overline{F_y})$. This states that the accumulation of wave activity depends on the convergence of the E-P flux.

One important application of the wave activity equation is the wave-mean flow interaction. We can easily show that the E-P flux also affects the mean flow. In the QG context, the zonal momentum equation

$$\partial_t u + \partial_x(\frac{u^2}{2}) + \partial_y(uv) - fv = P_x$$

can be zonally averaged to give

$$\partial_t \overline{u} + \partial_y \overline{u'v'} = 0$$

where we have used $\overline{v} = \overline{\partial_x \psi} = 0$ using the QG approximation. Notice (2.6.13), the equation above can be written as

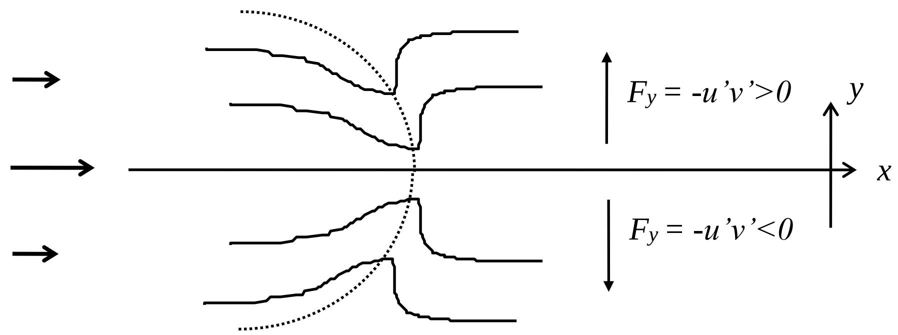
$$\partial_t \rho_0 D\overline{u} - \partial_y \overline{\boldsymbol{F}} = 0 \qquad (2.6.15)$$

or with (2.6.14),

$$\partial_t \rho_0 D\overline{u} + \partial_t \overline{A} = 0 . \qquad (2.6.16)$$

This suggests that the wave activity and mean flow exchange with each other. The wave-mean flow interaction occurs through the E-P flux convergence. (This can also be understood from the instability view later in Chapter 6: shear flow produces instability and the unstable wave feedbacks on the mean flow).

The above equations can be used to understand the negative viscosity. Consider a tilted wave in a sheared westerly jet,



In the middle of the jet, we have $-\partial_y F_y<0$ and therefore a decreasing wave activity $\partial_t A<0$. The decayed wave energy is converted to increase the mean jet, i.e. $\partial_t u > 0$ (in other words, because of the convergence of the Reynold stress). Therefore, in this case, the disturbance acts as a "negative viscosity" to the mean flow (Starr, 1950).

2. Wave activity for almost plane waves

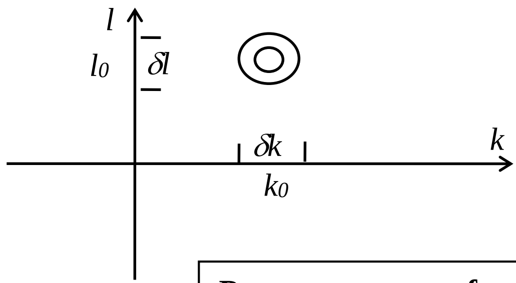
(1) Almost Plane Wave

As discussed in (2.6.2), a plane wave has a spectrum of a delta function. Now suppose the wave spectrum is finite but of a narrow band width,

$$\left|\delta k / k_0\right| << 1 \qquad \left|\delta l / l_0\right| << 1 \tag{2.6.17}$$

we have from (2.6.1) that

$$\psi' = \mathrm{Re}\left\{ \iint \hat{\psi}'(k+\delta k, l+\delta l) e^{i(\delta k x+\delta l y)} e^{-i\delta\omega_{k,l} t} d(\delta k) d(\delta l) \times e^{i(k_0 x + l_0 y)} e^{-i\omega_{k_0,l_0} t} \right\}$$
$$= \mathrm{Re}\left\{ \varphi(x,y,t) \times e^{i(k_0 x + l_0 y - \omega_0 t)} \right\}$$
$$= \mathrm{Re}\left\{ \varphi(x,y,t) \times e^{i\theta(x,y,t)} \right\}$$



Power spectrum of an almost-plane wave

where

$$\frac{\partial\theta}{\partial x} = k_0, \frac{\partial\theta}{\partial y} = l_0, \frac{\partial\theta}{\partial t} = -\omega_0$$
$$\frac{\partial\varphi}{\partial x} = i\delta k\varphi << k\psi \approx \frac{\partial\psi}{\partial x}$$

Thus, φ is slowly varying in *x, y, t*. The wave packet

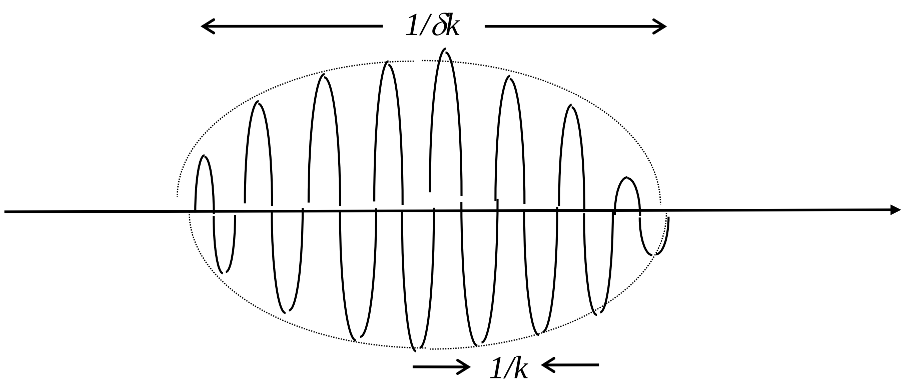
$$\psi' = \mathrm{Re}[\varphi(x,y,t)e^{i\theta}] \tag{2.6.18}$$

is called an almost-plane wave, with the slowly varying part φ as its envelope.

(2) WKB approximation: (Wentzel, Kramers, Brillouin) (or WKBJ: with Jeffreys)

Now, consider an almost plane wave,

$$\psi' = \mathrm{Re}\,\varphi(x,y,t)e^{i\theta} \quad \text{where} \quad \theta = kx + ly - \omega t$$



$$\partial_x \psi' = \text{Re}\{\partial_x \varphi e^{i\theta} + ik\varphi e^{i\theta}\} \approx k\,\text{Re}\, i\varphi e^{i\theta} = k\psi'[1 + O(\delta k / k)] \approx k\psi'$$

The linearized QGPV and its derivatives at the leading order is:

$$q' = (\nabla^2 - \frac{1}{L_D^2})\psi' \approx -(K^2 + L_D^{-2})\,\text{Re}\,\varphi e^{i\theta}$$

$$\partial_t q' \approx -\omega(K^2 + L_D^{-2})\,\text{Re}(i\varphi e^{i\theta}),$$

$$\partial_x q' \approx -k(K^2 + L_D^{-2})\,\text{Re}(i\varphi e^{i\theta}),$$

At the leading order, we have

$$\text{Re}\{[-i(K^2 + L_D^{-2})(\omega - kU) + ik\overline{q_y}]e^{i\theta}\varphi\} = 0$$

Thus,

$$\omega - kU = \frac{-kQ_y}{K^2 + L_D^{-2}} + O(\frac{\delta k}{k})$$

The dispersion relationship is therefore approximately

$$\omega = kU(y) - \frac{kQ_y(y)}{K^2 + L_D^{-2}} \equiv \Omega(y) \tag{2.6.19}$$

This is similar to the plane wave case, BUT, valid even in an inhomogeneous background flow $U = U(y)$, $Q_y = Q_y(y)$. The price, however, is that the relationship is no longer exact. In general, $\boldsymbol{k} = \boldsymbol{k}(y)$, so wave number is locally determined at *y* and could vary with *y*. To be consistent with slowly varying nature of ***k***, we need *U* and Q_y to be slowly varying in the sense that

$$\left|\frac{dU}{dy}\right| << |l(U - c)|, \text{ and } \left|\frac{d}{dy}Q_y\right| << |lQ_y|,$$

Or *U* and Q_y may vary on the scale of the wave packet $1/\delta k$ (WKB approximation) which is much larger than the wave scale $1/k$. We can also define a slowly varying local group velocity as:

$$\boldsymbol{C}_g = \left[\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\right] = \left[U + \frac{Q_y(k^2 - l^2 - L_D^{-2})}{(K^2 + L_D^{-2})^2}, \frac{2klQ_y}{(K^2 + L_D^{-2})^2}\right] = \boldsymbol{C}_g(y) \tag{2.6.20}$$

(3) Wave-activity of almost-plane waves:To derive the equation for the evolution of the wave packet, we could expand *x*, *y*, *t* in fast and slow variables (Pedlosky, Sec. 3.20). Here, instead, we use the more general wave activity equation (2.6.10): $\partial_t A + \nabla \bullet (\mathbf{U}A + \mathbf{F}) = S_A$

First, we define the phase average <> such that $< e^{i\theta} >= 0$. This is the average in the wave length scale, which is still much smaller than the wave packet scale. For example, $\langle\cos^2\varphi\rangle = \left\langle\frac{1}{2}(1 + \cos 2\varphi)\right\rangle = \frac{1}{2} + \frac{1}{2}\langle\cos 2\varphi\rangle = \frac{1}{2}$. This removes higher harmonics. Now, the wave activity is

$$A = A_0 + A_2 e^{2i\theta},$$

so,

$$\partial A=\partial A_0+Re(\partial A_2 -2i\omega A_2)e^{2i\theta},$$

$$<\partial_t A>=<\partial_t A_0>=\partial_t A_0=\partial_t <A>.$$

The wave activity equation can be written as

$$\partial_t <A>+\nabla\bullet\left[\boldsymbol{U}<A>+<\boldsymbol{F}>\right]=<S_A>,$$

$$<A>=\frac{\rho_0 D}{2}<q'^2>/Q_y,$$

Furthermore,

$$q'\approx-(K^2+L_D^{-2})\operatorname{Re}\varphi e^{i\theta}=-\frac{1}{2}(K^2+L_D^{-2})(\varphi e^{i\theta}+\varphi^* e^{-i\theta}),$$

$$<q'^2>=\frac{1}{4}(K^2+L_D^{-2})^2<\varphi^2 e^{2i\theta}+2\varphi^2+\varphi^{*2}e^{-2i\theta}>,$$

$$=\frac{1}{2}(K^2+L_D^{-2})|\varphi|^2,$$

Therefore,

$$<A>=\frac{\rho_0 D}{4}\frac{(K^2+L_D^{-2})^2}{Q_y}|\varphi|^2,$$

In general, for two complex variables $(a, b)=Re\,[(A, B)\,e^{i\theta}]$, we have $<ab>=Re(A,B^*)/2$. Therefore, we have

$$<F_x>=<\frac{\rho_0 D}{2}(v'^2-u'^2-\frac{\psi'^2}{L_D^{-2}})>$$

$$<v'^2>=<(\operatorname{Re}ik\varphi e^{i\theta})^2>=\frac{1}{2}k^2|\varphi|^2$$

$$<u'^2>=<(\operatorname{Re}il\varphi e^{i\theta})^2>=\frac{1}{2}l^2|\varphi|^2$$

$$<\psi'>^2=\frac{1}{2}|\varphi|^2$$

$$\boxed{<F_x>=\frac{\rho_0 D}{4}(k^2-l^2-L_D^{-2})|\varphi|^2}$$

$$<F_y>=-\rho_0 D<u'v'>$$

$$u'=-\operatorname{Re}(il\varphi e^{i\theta}),\quad v'=\operatorname{Re}(ik\varphi e^{i\theta})$$

$$<u'v'>=-\frac{kl}{4}<(i\varphi e^{i\theta}-i\varphi^* e^{-i\theta})^2>=-\frac{1}{2}kl|\varphi|^2$$

$$\boxed{<F_y>=\frac{\rho_0 D}{2}kl|\varphi|^2}$$

$$\mathbf{F} = (F_x, F_y) = \frac{<A> Q_y}{(K^2 + L_D^{-2})^2}(k^2 - l^2 - L_D^{-2}, 2kl) .$$

This reminds us of the local group velocity (2.6.20),

$$\boxed{<\mathbf{F}> = <A> (\mathbf{C}_g - \mathbf{U})} \tag{2.6.21}$$

The wave activity flux is the wave activity density transported by the group velocity relative to the mean flow!

Finally, for an almost-plane wave, the wave activity equation (2.6.10) becomes:

$$\boxed{\partial_t <A> + \nabla \bullet \left[\mathbf{C}_g <A>\right] = <S_A>} \tag{2.6.22}$$

If $\mathbf{C}_g$ is slowly varying, we further have

$$\partial_t <A> + \mathbf{C}_g \bullet \nabla <A> = <S_A>$$

This demonstrates that wave activity is transported by the group velocity speed. The concept of group velocity is the special case of almost plane waves. For more general cases, including those where the almost-plane wave concept fails, the corresponding concept of wave activity flux is always valid! This enables us to diagnose the wave activity flux in observations, which has complex variability and often violate the conditions for the almost-plane waves.

(4) Wave Activity and Wave Energy

What is the relation between the wave activity and wave energy? As in Section 2.1, the energy of a general perturbation is

$$E = \frac{1}{2}\rho D\left[u'^2 + v'^2 + \frac{\psi'^2}{L_D^{-2}}\right]$$

For the wave packet, the wave energy for almost-plane waves is therefore

$$<E> \approx \frac{\rho_0 D}{4}(K^2 + L_D^{-2})|\varphi|^2$$

$$= Q_y \frac{<A>}{(K^2 + L_D^{-2})} = (U - c) <A> \tag{2.6.23}$$

The wave activity equation for the almost plane wave (2.6.22) can be put in different forms.

(i) Conservation of wave action:

Since k is independent of y, multiplying $1/k$ in (2.2.22), we have

$$\boxed{\partial_t\left[\frac{<E>}{\omega - kU}\right] + \nabla \bullet \left\{\mathbf{C}_g \frac{<E>}{\omega - kU}\right\} = -\frac{1}{k} <S_A>} \tag{2.6.24}$$

where we have used (2.6.23) and the variable

$$\frac{< E >}{\omega - kU}$$

is called the wave action. The equation above therefore represents the conservation of wave action, which is the conservation of wave activity in the special case of almost-plane waves.

(ii) Wave energy equation:

Multiplying (2.6.23) by *U-c*, we have the wave energy equation

$$\partial_t < E > + \nabla \bullet \left[\mathbf{C}_g < E > \right] + \frac{\mathbf{C}_g \bullet \nabla \mathbf{U}}{U - c} < E > = (U - c) < S_A > \qquad (2.6.25)$$

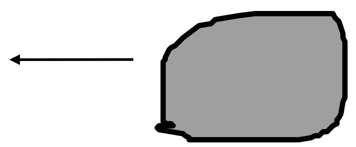
Therefore, wave energy is conserved ONLY IF the mean flow *U* is uniform or has no shear (relative to the direction of $\mathbf{C}_g$). In the presence of shear, there is energy exchange between the mean and the wave. The wave energy is no longer conserved! However, wave activity is still conserved, because it has already taken into account of wave-mean interaction.

(iii) Uniform basic state

If **U** and Q_y are uniform, ***k*** and $\mathbf{C}_g$ are uniform, so that

$$\partial_t < A > + \mathbf{C}_g \nabla \bullet < A > = < S_A >$$

In a uniform medium, wave activity (wave action and wave energy) of an almost plane wave packet all propagate at the group velocity. The wave packet will propagate without changing its shape. With an inhomogeneous medium, however, wave activity or wave action is a much more useful concept than the wave energy, the former tends to be conserved while the latter not.



(iv) Dissipation

With a dissipation, say a Rayleigh damping, $\mathbf{F} = -\lambda \rho \mathbf{u}'$, we have

$$curl \frac{\mathbf{F}}{\rho} = -\lambda \nabla^2 \psi', \qquad S_A = D\rho_0 q' \nabla^2 \psi' / Q_y .$$

For an almost -plane wave (2.6.18), we have

$$curlF = \lambda K^2 \operatorname{Re} \varphi e^{i\theta},$$

$$q' = (\nabla^2 - L_D^{-2})\psi' \approx -(K^2 + L_D^{-2}) \operatorname{Re}(\varphi e^{i\theta}),$$

$$< S_A >= \frac{D\rho_0}{\bar{q}_y} < q'\nabla^2\psi' >= -D\rho_0 \frac{\lambda}{2} \frac{K^2(K^2 + L_D^{-2})}{\bar{q}_y} |\varphi|^2 = \frac{-2\lambda K^2}{K^2 + L_D^{-2}} < A > .$$

Thus, the wave activity equation (2.6.22) can be written as:

$$\partial_t < A >= -\nabla \bullet (\mathbf{C}_g < A >) - \frac{2\lambda K^2}{K^2 + L_D^{-2}} < A > .$$

The effective damping rate is therefore:

$$\frac{2\lambda K^2}{K^2 + L_D^{-2}} \xrightarrow[\text{baratropic Limit } (L_D^2 \to \infty)]{} 2\lambda$$

Exercises for Chapter 2

E2.1: (Rossby wave pattern) (a) The perturbation streamfunction of the Rossby wave is $\psi'(x,t)=cos(kx-\omega t)$. What is the wave pattern? (b) The mean zonal flow is U=const. What is the pattern of the total streamfunction $\psi(x,y,t) = -Uy + \psi'(x,t)$?

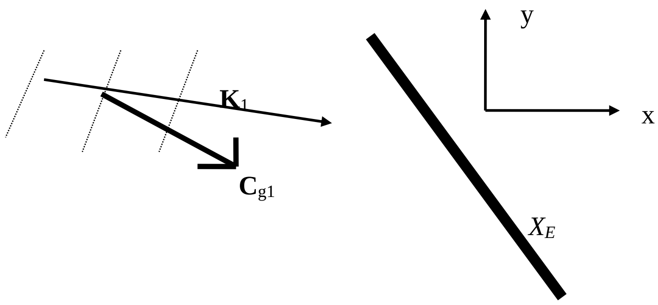
E2.2: (Wave envelop and group velocity) Two plane Rossby waves have similar wave numbers such that the streamfunctions are $\psi_1=cos(kx-\omega t)$ and $\psi_2=cos[(k+\Delta k)x-(\omega+\Delta\omega)t]$, where $\Delta k<<k$. (a) Show that the total streamfunction $\psi=\psi_1+\psi_2$ consists of a Rossby wave of wave number close to k and an wave envelope with the wave number Δk. (b) What is the propagation speed of the each individual wave crest (phase speed)? (c) What is the propagation speed of the wave envelope (group velocity)?

E2.3: (Propagation of planetary wave) For a Rossby wave whose wave length is much longer than the deformation radius (planetary wave), relative vorticity is negligible such that the potential vorticity is approximately $q=f/h$. In this case, why does the planetary wave propagate westward? Illustrate the propagation mechanism based on the principle of PV conservation.

E2.4: (Dispersion diagram) In the absence of mean flow and bottom topography, (a) Calculate the group velocity of the Rossby wave $\boldsymbol{C_g}=(C_{gx},C_{gy})$. (b) Verify the Rossby wave dispersion diagram in Sec.2.3.

E2.5: (Group velocity) Given the dispersion relationship $\omega=-\beta k/(k^2+l^2+L_D^{-2})$ and a specific meridional wave number l, find the maximum eastward and westward group velocities as well as the wave number at which each maximum is achieved.

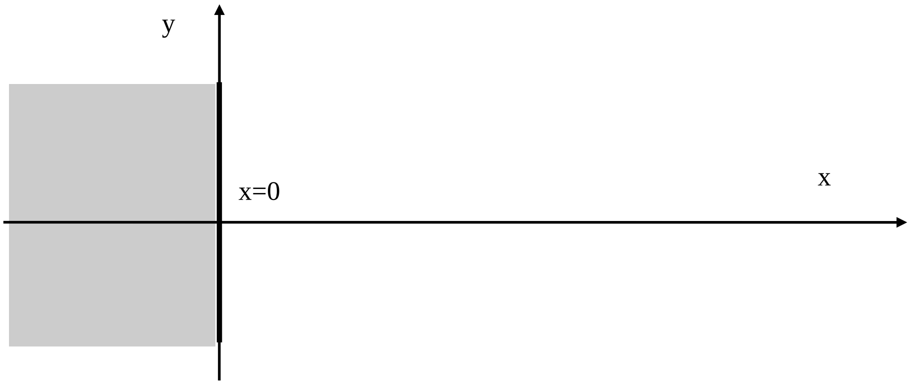
E2.6: (Rossby wave reflection) An incident Rossby wave $\psi_1=A_1 exp[i(k_1x+l_1y-\omega_1 t)]$ impinges on a tilted eastern boundary $x_E(y)=y/a$, where a is a constant. Identify the incident and reflected waves on the dispersion diagram.



E2.7: (Coastal Kelvin wave) On a half-plane $x>0$, shallow water disturbances satisfy

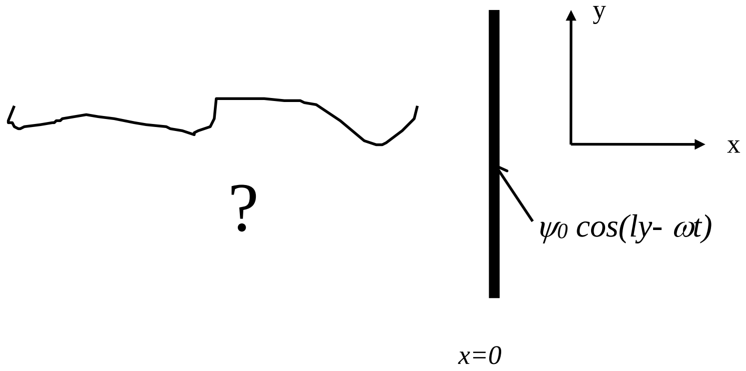
$$\partial_t u-fv = -g\partial_x\eta,\ \partial_t v+fu = -g\partial_y\eta,\ \partial_t\eta+H(\partial_x u+\partial_y v) = 0,$$

in $x > 0$. The flow has to satisfy the solid wall boundary condition $u|_{x=0} =0$. In addition, there is no energy source from the infinity, so the energy radiation condition requires finite u, v, η at $x\to+\infty$.

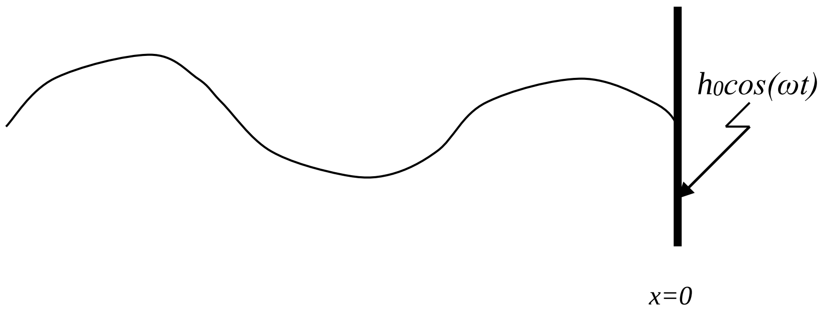


Assume this is an f-plane, find the linear wave solution.(Hint: set u=0 as part of the solution).

E2.8: (Radiating Rossby wave): A wavemaker on the eastern boundary $x=0$ has a frequency ω and a meridional wave number l, such that the surface elevation on $x=0$ satisfies $\eta=\eta_0 cos(ly-\omega t)$. The wavelength is sufficiently long (large scale) and the frequency is sufficiently low such that quasi-geostrophic dynamics can be applied. Find the Rossby waves generated by this wavemaker. (Hint: find the Rossby wave that satisfies the eastern boundary condition $\psi=\psi_0 cos(ly-\omega t)$ at $x=0$, where $\psi_0=g\eta_0/f_0$.



E2.9. (Radiating gravity wave) On a nonrotating tank, the shallow water waves satisfy the equations: $\partial_t u=-g\partial_x h-ru$, $\partial_t h+H\partial_x u=-rh$, where r is the damping rate. There is a wavemaker on the eastern boundary $x=0$, such that the coastal sea level is forced to oscillate as $h(x=0,t)=h_0\cos(\omega t)$. Find the sea level response in the interior ocean for two cases. (a) No damping such that $r=0$, (b) finite damping with $r>0$. In each case, discuss the physics of the solution.



E2.10. (Transient resonant solution): A forced swing set satisfies the equation

$dA/dt-i\omega A=F\exp(i\sigma t)$, where A is the position of the swing and ω is the frequency of the free oscillation, σ is the forcing frequency and F is the forcing amplitude. The forcing amplitude F is applied at t=0. Derive the forced solution that satisfies the initial condition: $A(t=0)=0$ for (a) $\sigma\neq\omega$ and (b) $\sigma=\omega$. [Hint for b): try the solution of the form $A=C_1\exp(i\omega t)+C_2 t\exp(i\omega t)$ and determine the coefficients C_1 and C_2].

E2.11: (Rossby wave energy flux) A free Rossby wave has the form $\psi=Re[A\, e^{\,i(kx+ly-\omega t)}]$

and the dispersion relationship $\omega=-\beta k/(k^2+l^2+L_D^{-2})$.

a) Prove that the wave energy (averaged within a wave length) is:

$$E=<\{\psi_x^2+\psi_y^2+(\psi/L_D)^2\}/2>=|A|^2(k^2+l^2+L_D^{-2})/4$$

b) Prove that the group velocity can be written as:

$$(C_{gx}, C_{gy})=[k+\beta/(2\omega),\ l]\times(-2\omega)/(k^2+l^2+L_D^{-2}).$$

The energy flux of a wave packet $\mathbf{F}=(F_x, F_y)$ is the wave energy multiplied by the group velocity $\mathbf{F}=E\times(C_{gx}, C_{gy})= -2\omega|A|^2(k+\beta/2\omega,\ l)$.

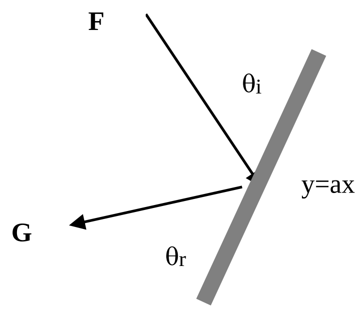
c) If this wave is reflected on an eastern boundary $y=ax$ (see ***E2.6***), prove that the reflected wave $\phi=Re[B\, e^{\,i(mx+ny-\sigma t)}]$ has the same alongshore energy flux as the incident wave, that is $\mathbf{G}\bullet(1, a) = \mathbf{F}\bullet(1, a)$, where $\mathbf{G} = (G_x, G_y)$ is the energy flux of the reflected wave.

d) Are the alongshore group velocities the same for the incident and reflected waves?

e) Prove that the incident angle θ_i is

$$\cos(\vartheta_{\mathrm{i}}) = \frac{\mathbf{F}\bullet(1,\mathrm{a})}{|\mathbf{F}|\times|(1,\mathrm{a})|} = \frac{\mathbf{F}\bullet(1,\mathrm{a})}{\sqrt{(\frac{\beta}{2\omega})^2 - L_D^{-2}}\times\sqrt{1+a^2}}$$

f) A similar expression can be derived for the reflection angle. Is the incident angle the same as the reflection angle?



E2.12 (Topographic Rossby Wave): On a f-plane with homogeneous fluid, a Northern Hemisphere ocean basin (Fig.E2.12.1: a. top view, b. side view) has a continental slope of several hundred kilometers wide. For large scale, low frequency disturbances,

a) What kind of low frequency waves will be produced in the basin? Which direction does it propagate? State clearly the physical mechanism for this wave and how it propagates in the direction you proposed. What will be the equation that controls the basic dynamics of this wave?

b) What will happen if this basin is in the Southern Hemisphere (Fig.E2.12.2)?

c) What happens around a large scale seamount in the Northern Hemisphere (Fig.2.12.3)?

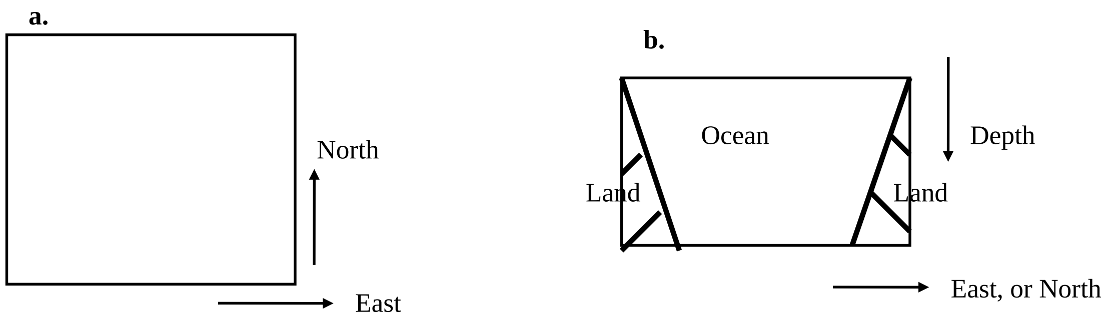


Fig.2.12.1: NH basin with continental slope, $f>0$

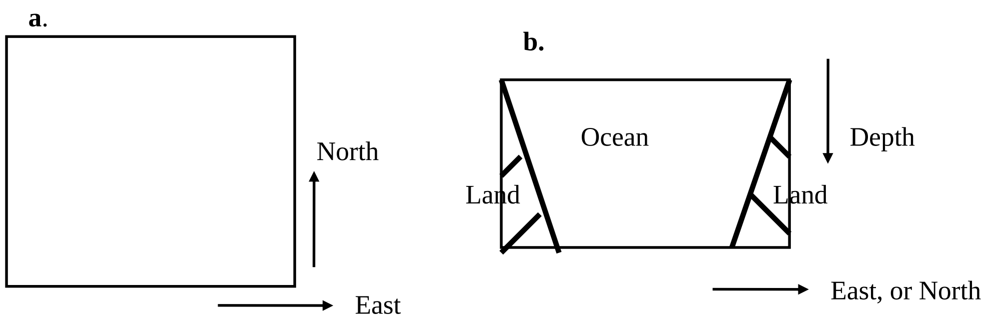


Fig.E2.12.2: SH basin with continental slope, $f<0$

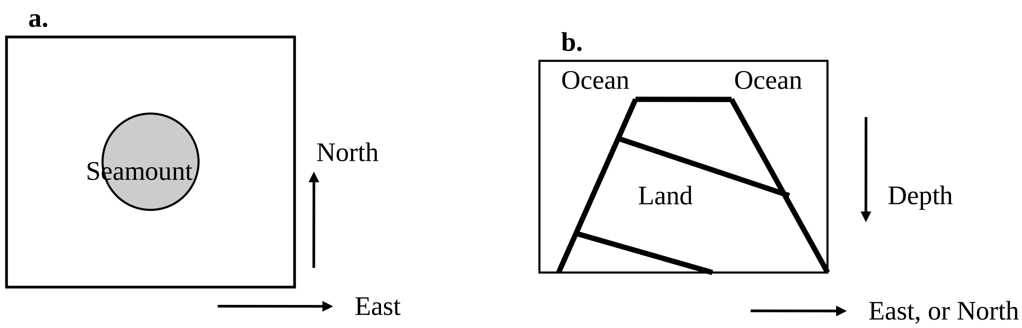


Fig.E2.12.3: NH Seamount, $f>0$