Ch.3: Forced General Circulation

Sec. 3.1: Atmospheric Circulation

On the horizontal plane (or isobar, isotropic surface), the atmospheric circulation is relatively simple. At the leading order, it consists mainly of zonal flows. At the next order, the zonal flow is distorted by stationary waves that are forced by localized surface heating (due to land-sea contrast) and topography. (see Figs.3.1, 3.2). One may think that the reason for the first order flow to be dominantly zonal $U(y)$ is due to the solar radiation that is zonally uniform. This is only half true.

The other fundamental reason for the presence of a dominant zonal flow $U(y)$ is the rotation, or more precisely the beta effect. The rotation of the earth restrains the fluid to flow along constant potential vorticity lines (latitude circle) in the absence of strong damping and diabatic heating. Furthermore, the absence of meridional boundaries allows the $U(y)$ flow to be free modes in the atmosphere (of zero frequency). These "free modes", when forced by the zonally uniform heating, becomes dominant due to resonant excitation.

The effect of beta can be seen easily in the context of the QG equation. The PV conservation gives the equation for the steady state circulation as:

$$J(\Psi, Q)=0, \tag{3.1.1}$$

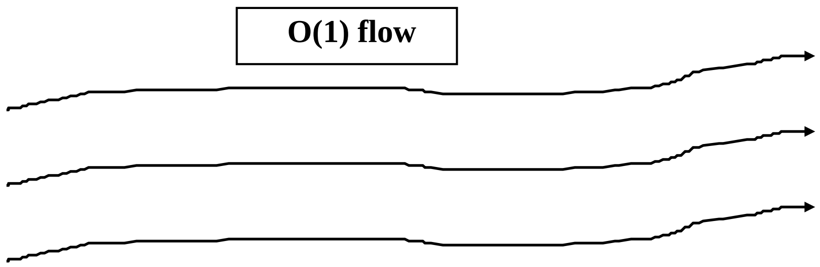
or

$$\Psi= \Psi(Q)$$

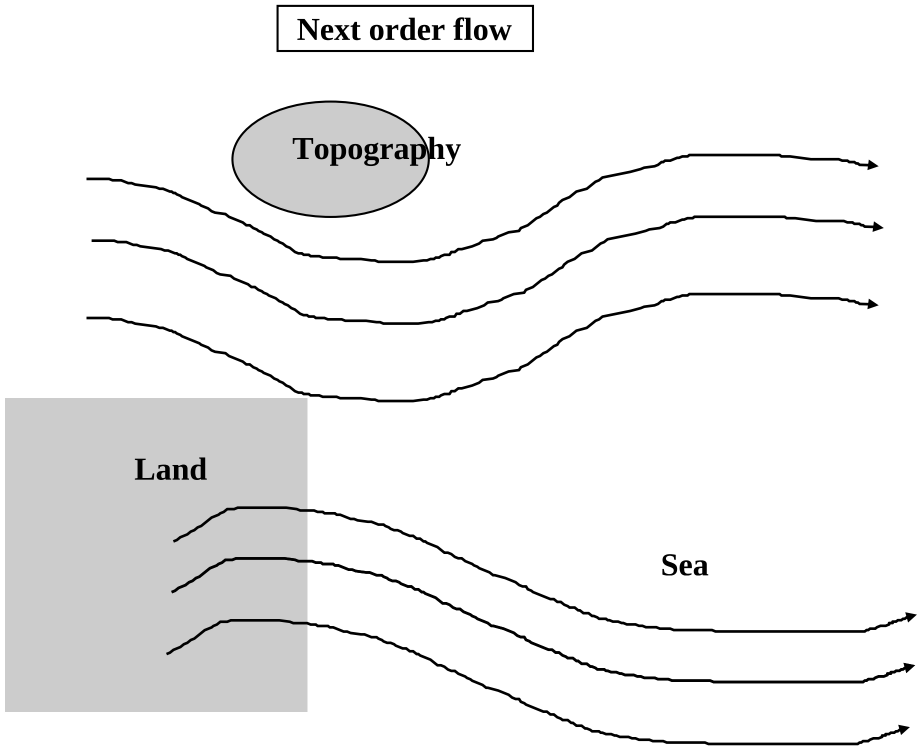
At the first order, the potential vorticity is $Q=\beta y$. Thus, we have

$$\Psi= \Psi(\beta y) =G(y). \tag{3.1.2}$$

This is a zonal flow, a free mode of the equation of motion.



At the next order, the effect of localized topography or land-sea contrast enters and the zonal flow is distorted. The distortion is the strongest in the surface layer (Fig.3.3), because of the strong dissipation and therefore the less efficient excitation of the zonal flow free mode.



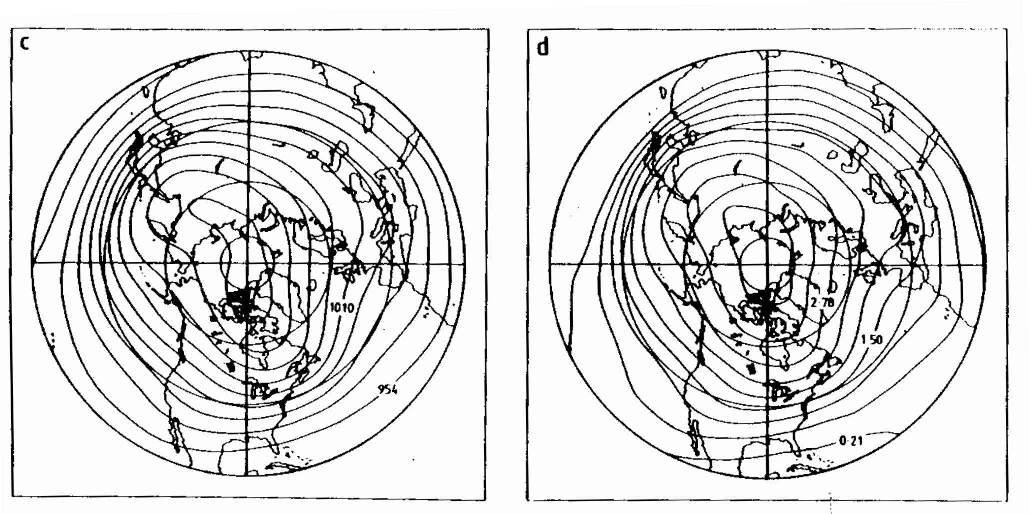


FIG. 1. Time-averaged structure of the Northern Hemisphere winter flow. Mean geopotential height (dam) at (a) 850 mb, (b) 500 mb and (c) 250 mb; (d) Mean quasi-geostrophic potential vorticity (10^{-4} s^{-1}) at 250 mb.

Fig.3.1 Atmospheric circulation in the middle and upper levels.

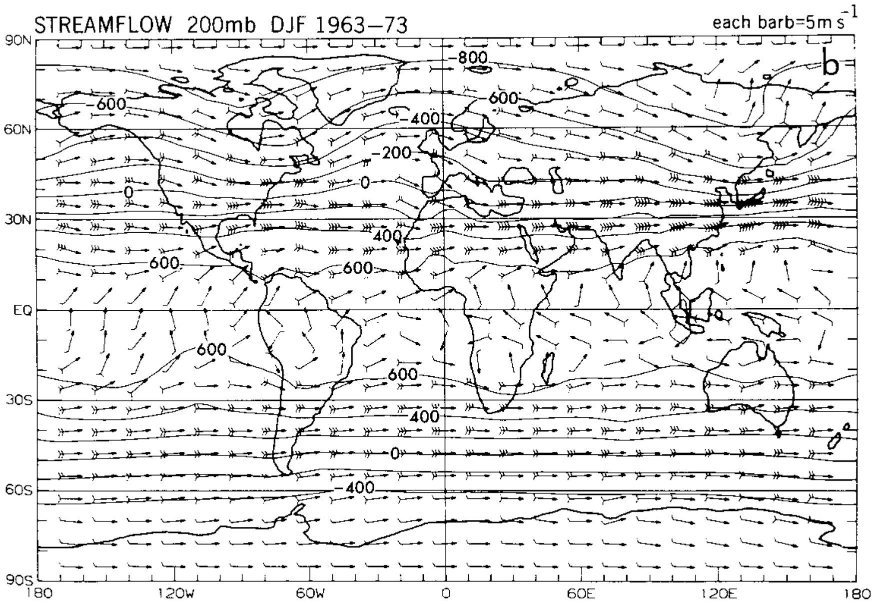


FIGURE 7.13b

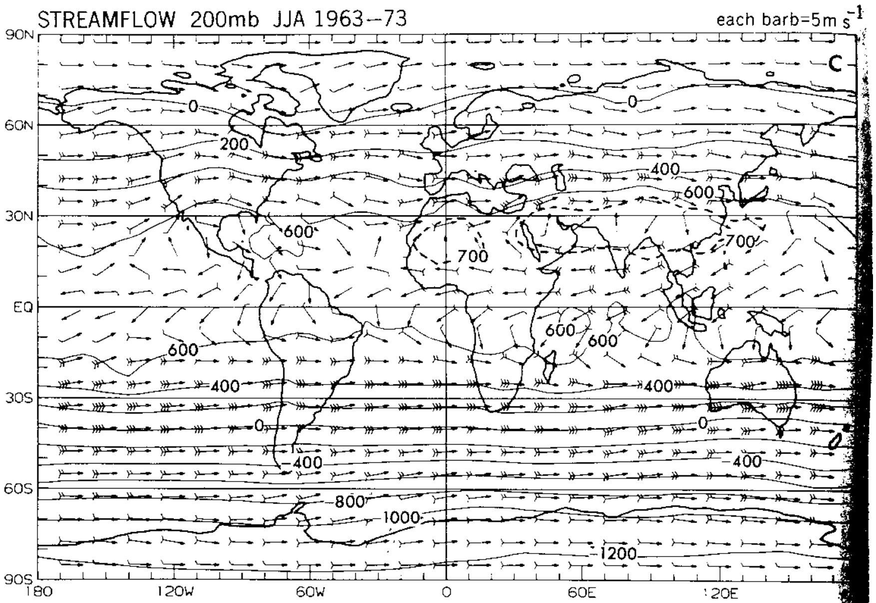


FIGURE 7.13. Global distributions of the height of the 200-mb pressure field, $\bar{z}_{200} - 11$ gpm, in gpm for annual-mean (a), northern winter (b), and northern summer (c) mean conditi Also shown are vector plots of the 200-mb winds. For geostrophic flow the arrows should par the isolines. Each barb on the tail of an arrow represents a wind speed of 5 m s^{-1}.

Fig.3.2 Atmospheric circulation on the 200mb

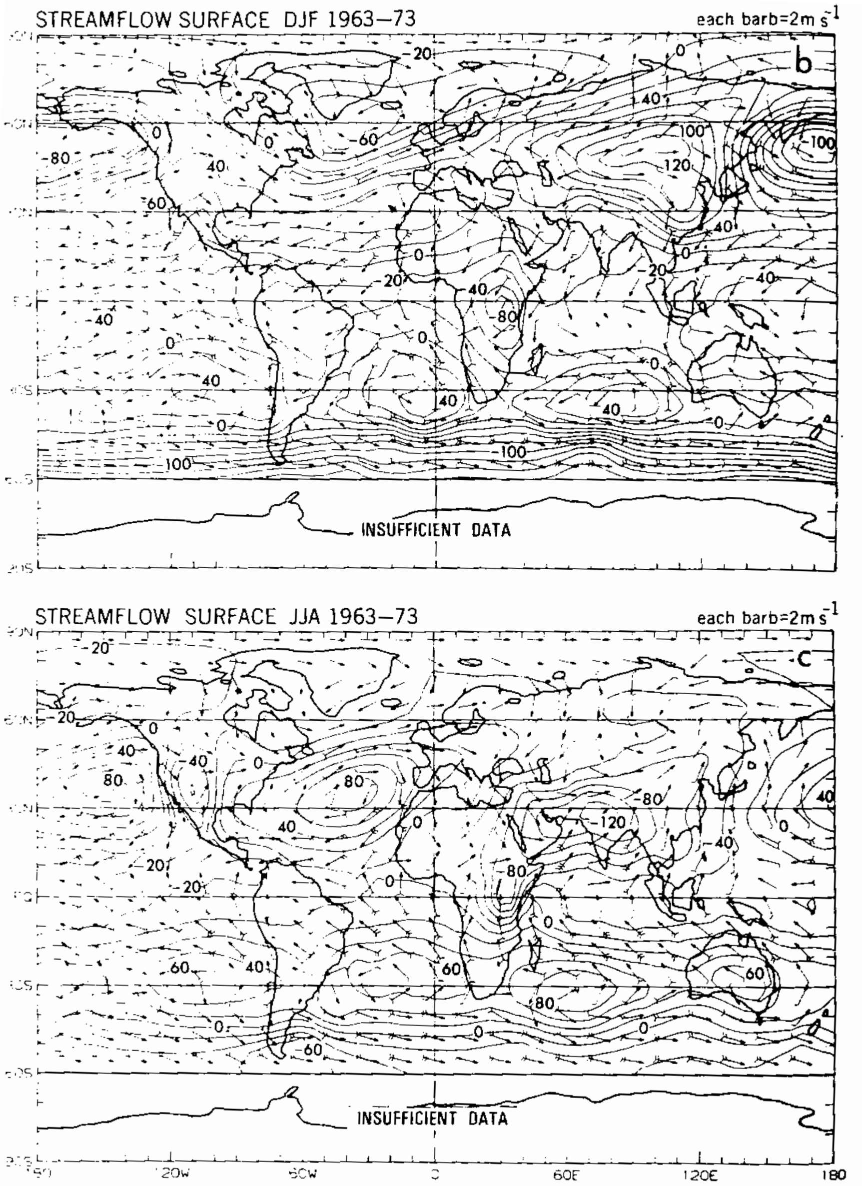


FIGURE 7.1. Global distributions of the height anomalies of the 1000-mb pressure field, $\bar{z}_{1000} - z^{SA}_{1000}$, in gpm for annual-mean (a), northern winter (b), and northern summer (c) mean conditions. The quantity z^{SA}_{1000} ($= 113$ gpm) represents the height of the 1000-mb level according to the NMC standard atmosphere. Also shown are vector plots of the surface winds. For geostrophic flow the arrows should parallel the isolines. Each barb on the tail of an arrow represents a wind speed of 2 m s^{-1}. The isoheight lines can also be interpreted as isobars for the surface pressure reduced to sea level. Since 1 gpm is equivalent to about 0.121 mb, we can, e.g., relabel an isoheight line of $+40$ gpm by $(40 + 113)\times 0.121 + 1000 = 1018.4$ mb, and an isoheight line of -40 gpm by $(-40 + 113)\times 0.121 + 1000 = 1008.8$ mb. Note that $1 \text{ gpm} \approx 1$ m.

Fig.3.3 Atmospheric circulation near the surface

Sec. 3.2: Ekman Flow, Ekman Layer and Ekman Pumping

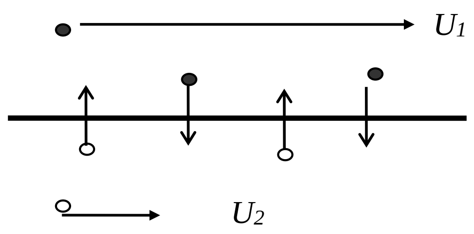
In comparison to the atmosphere, the general circulation is much more complex in the ocean. Most importantly, the circulation is no longer dominated by a zonal flow. Instead, it consists of huge gyres with flows comparable in both the zonal and the meridional directions. The most important reason for the difference of the circulations of the atmosphere and ocean is the presence of meridional boundaries. These meridional boundaries block the mean PV isoline (which is mostly around the latitude circle). As a result, the zonal flow (3.1.2), although satisfying the equation (3.1.1), does not satisfy the boundary condition anymore; they are no longer free modes and therefore can't be excited resonantly as in the atmosphere. In the ocean, therefore, the circulation at the leading order is forced by a strong momentum and buoyancy flux forcing. In the following, we first study how the surface ocean is forced by the wind stress.

1. Mixing and Friction

The wind stress forces the ocean through momentum mixing at the surface In general, fluid layers with different velocity have interfacial momentum exchanges due to cross-interface turbulences. These turbulence exchange heat and other water properties. The turbulences are usually strong near the boundary layer where the shear or gradient is the strongest. We will take the stress and the associated momentum flux as an example.

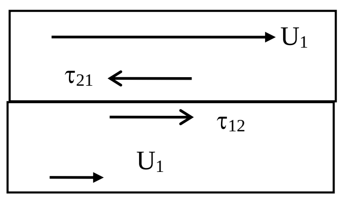
(1) Stress and momentum flux

Consider two layers of fluid with different mean velocities, say, faster in the upper layer



Because of random turbulent mixing of water parcels across the interface, the lower layer transports slower water parcels to the upper layer. These slower parcels slow down the upper layer and therefore appears as an interfacial drag stress that decelerates the upper layer. To conserve mass, the upper layer transports the same amount of parcels that are faster than the lower layer back to the lower layer. These faster parcels increase the mean velocity in the lower layer and therefore act as a driving stress (such as the wind stress). This interfacial stress is the Reynold's stress. Denoting τ_{nm} as the stress of layer n on layer m, we have

$$\tau_{12} = \tau \downarrow = -\overline{u'w'} = -\tau \uparrow = -\tau_{21}$$



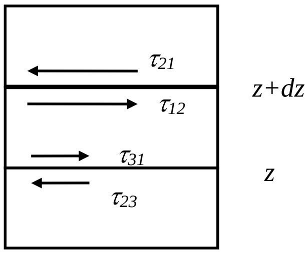
The simplest way to parameterize the stress is to realize that the interfacial stress is usually proportional to the shear of the mean velocity $\tau \approx \frac{\partial U}{\partial z}$. Thus at the lowest order, the stress is parameterized, in the so called *K* theory, as

$$\tau_{12} = -\overline{u'w'} = k\frac{\partial U}{\partial z} \tag{3.2.1}$$

(An example is in Section. 2.6 on the negative viscosity associated with $-\overline{u'v'}$)

(2) Net Effect

The net effect of the stress on a body of fluid is determined by the shear of the stress. In the figure, the net effect is:



$$\rho \Delta z \frac{\partial u}{\partial t} = \tau_{12} - \tau_{23}$$

or in the limit of infinitesimal Δz

$$\rho\frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial z} \tag{3.2.2}$$

Using the K-theory parameterization, we have the effect of the stress as

$$\frac{\partial u}{\partial t} = ...+\frac{1}{\rho}\partial_z(k\partial_z u) = ...\partial_z(d\partial_z u) \tag{3.2.3}$$

where $d=k/\rho$ is the kinematic diffusivity. In the simplest case of a constant ρ, (3.2.3) reduces to the standard diffusive equation

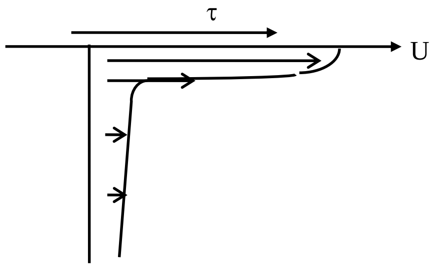
$$\partial_t u = d\partial_{zz}u \tag{3.2.4}$$

Similar idea can be applied to other directions and give the more general diffusive equation $\partial_t u = k_1\partial_{xx}u + k_z\partial_{yy}u + k_3\partial_{zz}u$. Furthermore, similar idea can be applied to other water properties such as the temperature, resulting in the diffusive equation for temperature as $\partial_t T = k\partial_{zz}T$.

2. Ekman flow

Now, we consider how the wind stress forces the ocean. In a very thin layer at the air-sea interface, ∂_{zz} is very large where the wind stress is very important by directly inputting momentum flux into the ocean. For large scale, low frequency flow, we have

$$-fv = -g\eta_x + \frac{\partial_z F^x}{\rho}, \quad fu = -g\eta_x + \frac{\partial_z F^y}{\rho}$$



There are two forcing here, the pressure gradient force and the wind stress force. To focus on the wind driven flow, we only consider the net wind driven flow

$$\vec{v}_E = \vec{v} - \vec{v}_g$$

where $\vec{v}_g = (-g\partial_y\eta, g\partial_x\eta)$ is the geostrophic flow.

Notice $\partial_z v_g = 0$, we have

$$-fv_E = \frac{\partial_z F^x}{\rho}, \quad fu_E = \frac{\partial_z F^y}{\rho} \tag{3.2.5}$$

The boundary conditions are:

$$\frac{F^x}{\rho} = \frac{\tau^x}{\rho}, \quad \frac{F^y}{\rho} = \frac{\tau^y}{\rho} \qquad \text{at } z = 0 \tag{3.2.6a}$$

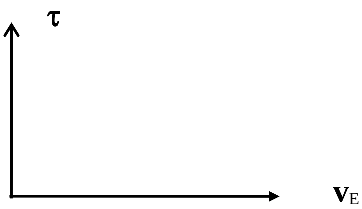
$$\frac{F^x}{\rho} \to 0, \quad \frac{F^y}{\rho} \to 0 \qquad \text{at } z \to -\infty \tag{3.2.6b}$$

where (τ^x, τ^y) are the surface wind stress .

Before solving the equation, we consider the vertically integrated flow (transport)

$$V = \int_{-\infty}^{0} v dz$$

Vertical integration of (3.2.5) gives



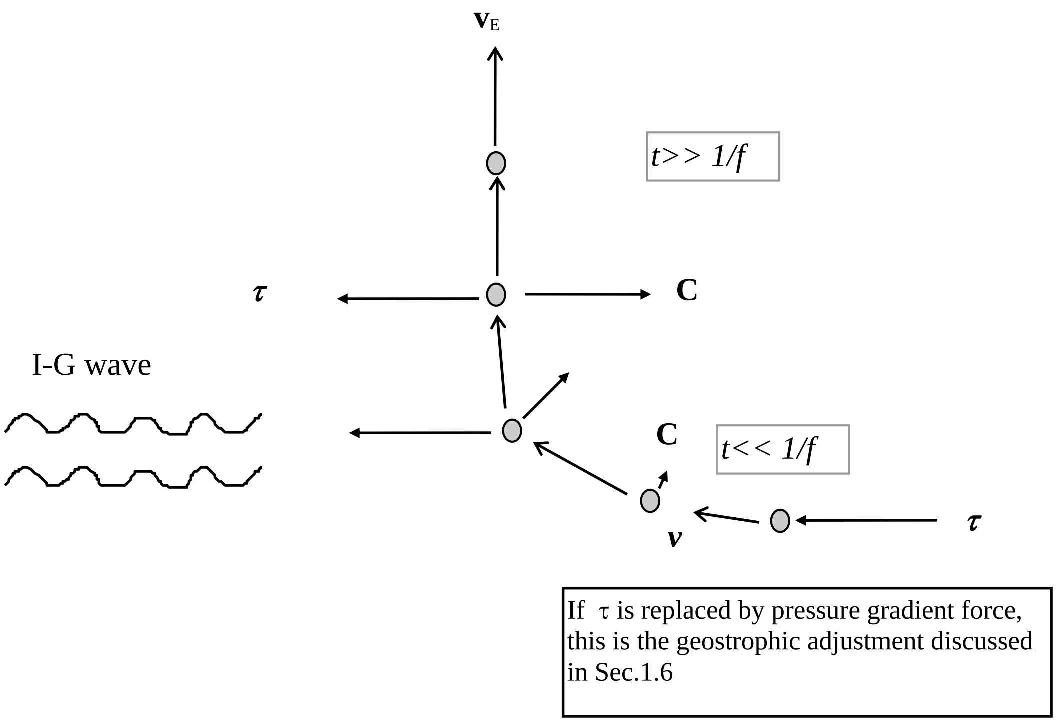
$$fV_E = -\frac{\tau^x}{\rho}, \quad fU_E = \frac{\tau^y}{\rho}$$

or

$$\mathbf{V}_E = (U_E, V_E) = (\frac{\tau^y}{\rho f}, -\frac{\tau^x}{\rho f}) = \frac{\boldsymbol{\tau} \times \mathbf{k}}{\rho f} \tag{3.2.7}$$

The wind driven flow is always to the right of the wind! This is called the Ekman flow or Ekman transport.

One can think of the spin-up process similar to the geostrophic adjustment process, except to replace the pressure gradient force by the wind stress. In this sense, Ekman flow and geostrophic flow are the same, but for different driving force.



3. Ekman Layer

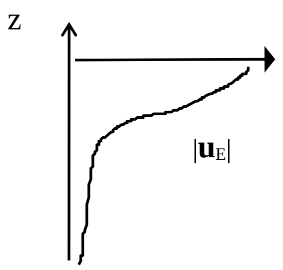
In spite of the simplicity of the vertically integrated flow, the current is more complex within the thin boundary layer, or the so called Ekman layer. Using the K-theory of parameterization

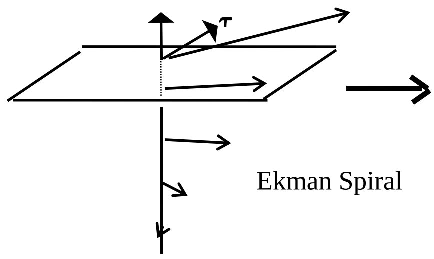
$$\frac{F^x}{\rho} = d\partial_z u, \quad \frac{F^y}{\rho} = d\partial_z v\,, \tag{3.2.8}$$

and the boundary conditions (3.2.6), the solution to eqns. (3.2.5) can be solved explicitly in the case of a constant *d*. With a wind stress of $\boldsymbol{\tau} = (\tau, 0)$, the Ekman layer velocity can be written as:

$$u_E + iv_E = \frac{\delta_e}{2\rho d}(1+i)\tau e^{\frac{z}{\delta_z}}\left[\cos\left(\frac{z}{\delta_z}\right) + i\sin\left(\frac{z}{\delta_z}\right)\right] \tag{3.2.9}$$

The surface current is 45^o to the right of the wind stress. The velocity field rotates clockwise with depth in the northern hemisphere. The e-folding decay scale with depth is $\delta_e = \sqrt{\frac{2d}{f}} \sim 50m$ in the ocean and 500m in the atmosphere.





It is important to realize that although the detailed structure of the Ekman layer depends on the mixing parameter (3.2.8), the vertically integrated Ekman transport $\mathbf{V}_E$ in (3.2.7) is independent of mixing parameterization. This is the beauty of the Ekman theory!

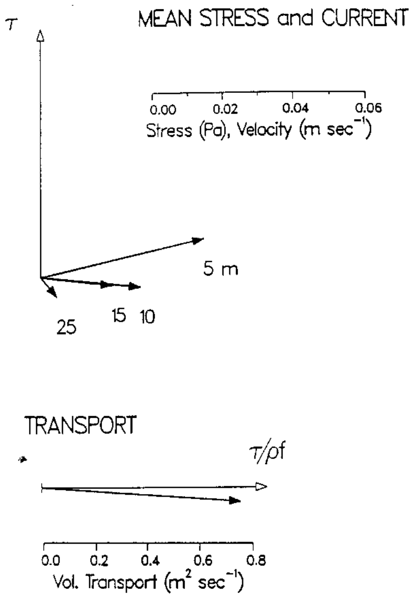


FIGURE 2.8. Wind-coherent mean wind stress, current spiral and transport relative to 50 m from the LOTUS 3 observations. The wind stress vector is arbitrarily pointed up, or "north", and is shown with an open arrowhead; current vectors are shown with solid arrowheads. Numbers at the ends of the current vectors are the depth in meters. The theoretical Ekman volume transport equal to $\tau/\rho f$ is shown with an open arrowhead, and the observed transport relative to 50 m is shown with a solid arrowhead. Uncertainties are given in Tables 2.1 and 2.2. As described in the text, the 5-m current is an 86-day average, the 15 m current is a 149-day average, and 10 and 25 m currents are 160-day averages.

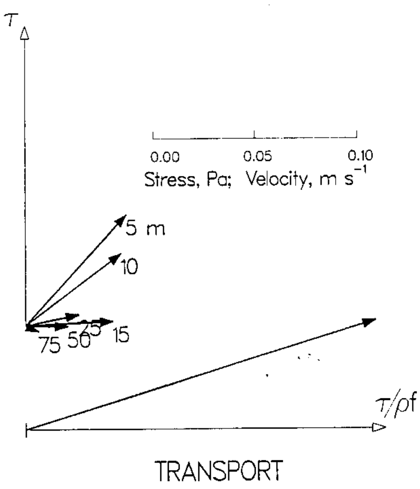
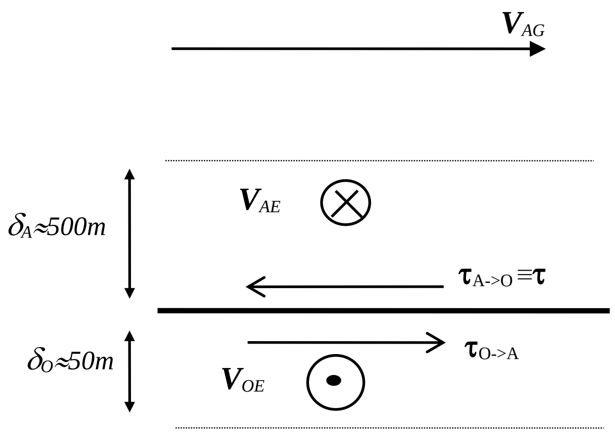


FIGURE 3.3. Mean current spiral and transport during the LOTUS 4 period. Uncertainties are given in Tables 3.1 and 3.2. The current spiral has a large downwind shear in the upper 15 m, and the shear abruptly changes to the crosswind direction below 15 m. The observed transport agrees with the predicted Ekman transport except for the large downwind component. As described in the text, the 75-m current is a 50-day average, the 25 m current is a 99-day average, and all other currents are 109-day averages.

Fig.3.4 Observed Oceanic Ekman layer

4. Atmospheric Ekman Flow

In the atmosphere, there is also an Ekman flow in the opposite direction as the oceanic Ekman flow. This is because, when the wind stress forces the ocean, the ocean exerts an opposite stress on the atmosphere. This ocean-to-atmosphere stress is the retarding stress that makes atmosphere wind aloft vanish on the sea surface.



Thus the Ekman flow in the atmosphere is

$$\mathbf{V}_{AE} = \frac{\boldsymbol{\tau}_{O->A} \times \mathbf{k}}{f\rho_A} = -\frac{\boldsymbol{\tau} \times \mathbf{k}}{f\rho_A}$$

$$= -\mathbf{V}_{OE} \frac{\rho_O}{\rho_A}$$

or

$$\rho_A \vec{V}_{AE} = -\rho_O \vec{V}_{OE}$$

Therefore, the mass flux is the same in both Ekman layers. Since the atmospheric density is about 1000 times smaller than that of the ocean, the velocity of the Ekman flow must be about 1000 times faster in the atmosphere than that in the ocean.

$$\frac{\rho_A}{\rho_O} \propto \frac{1}{1000} \quad \rightarrow \quad \left|\frac{V_{OE}}{V_{AE}}\right| \propto \frac{1}{1000}.$$

Observations show that

$$V_{AE} \approx 10\, m/s \quad *1000m, \quad V_{OE} \approx 10\, cm/s \quad *100m,$$

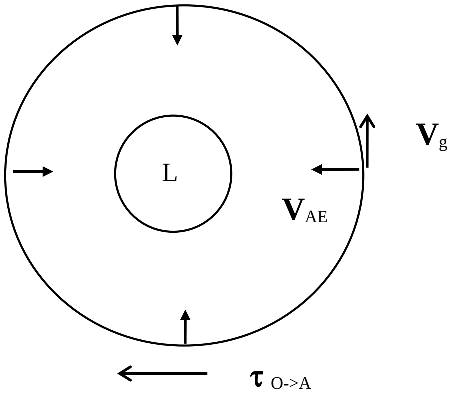
are at the same order of our estimation.

An interesting application is the heat flux

$$\frac{\rho_A C_{pA} V_{AE} T_A}{\rho_O C_{pO} V_{OE} T_O} \approx \frac{C_{pA}}{C_{pO}} \approx \frac{1}{4}$$

This states that the meridional heat transport is about four times larger in the ocean than in the atmosphere. This is comparable with the observational estimation in the low latitude, where both V_{ae} and V_{oe} are the main heat transport mechanisms. (in the mid- and high latitudes, atmospheric eddies and ocean boundary currents could play critical roles).

In addition, in the meteorological convention, V_{AE} is always down the pressure gradient of the geostrophic wind field.

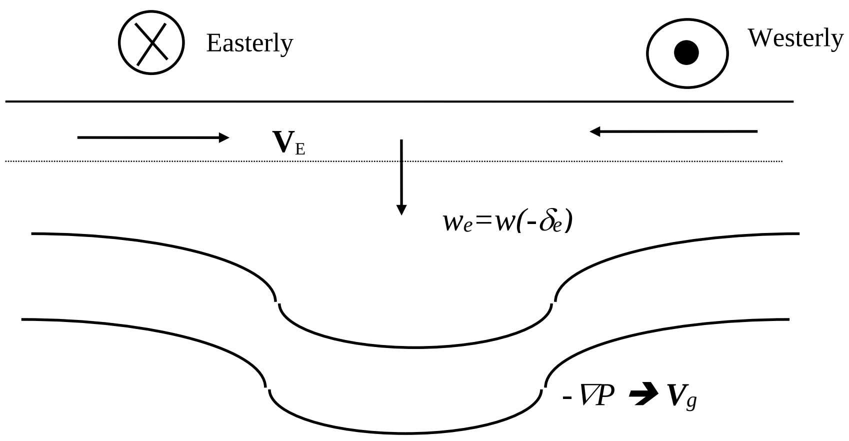


This is because in the boundary layer, the friction effect balances part of pressure gradient force. This atmospheric Ekman flow can be understood as the atmospheric response to the oceanic stress on the atmosphere. The latter is the opposite to the atmospheric wind stress on the ocean.

5. Ekman Pumping

If the wind stress forces Ekman flow only within the Ekman layer, does this mean the wind stress only drives the ocean current within the surface Ekman layer (about δ_e~50m)? A similar question in the atmosphere is that, if the Ekman flow is confined to δ_e, does this mean the bottom friction has no effect on the air above δ_e? The answer is: No!

Although the direct momentum mixing due to wind is limited within δ_e, the Ekman flow may produce vertical motion that penetrates deep into the subsurface ocean. For example, in a subtropical gyre,



The convergence of the Ekman flow produces a downward mass pumping into the subsurface ocean. The vertical motion distorts the subsurface pressure field to produce horizontal pressure gradient in the subsurface. The resulting pressure gradient force then forces geostrophic current through the geostrophic adjustment process. Therefore, the wind can force the subsurface flow, albeit indirectly. The vertical motion produced is called the Ekman Pumping.

The magnitude of the Ekman pumping can be obtained by integrating on the continuity equation in the Ekman layer as follows.

The continuity equation

$$\partial_x u + \partial_y v + \partial_z w = 0$$

can be written as

$$\partial_x u_e + \partial_y v_e + \partial_z w = 0$$

because the geostrophic flow is non-divergent. Integrating this in the Ekman layer

$$\int_{\delta_e}^{0} dz = \partial_x U_e + \partial_y V_e + w(0) - w_e = 0 \qquad (3.2.9)$$

where $w_e = w(-\delta_e)$ is the Ekman pumping velocity.

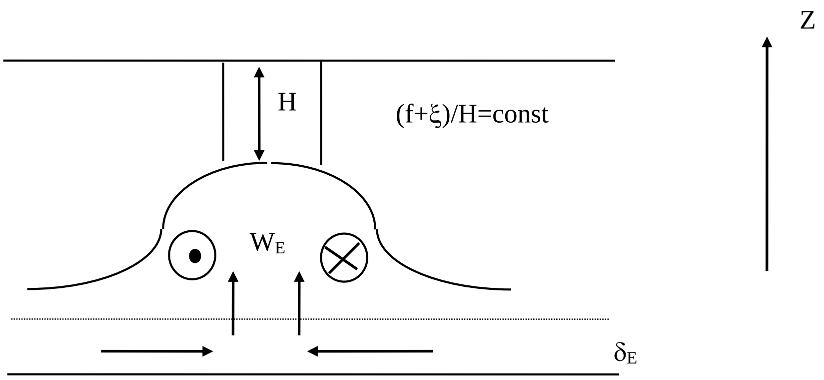
<u>Example 1:</u> Oceanic wind driven circulation:

Using a surface rigid lid *w(0)=0*, the Ekman pumping velocity is

$$w_e = \partial_x U_e + \partial_y V_e = \partial_x\left(\frac{\tau^y}{\rho f}\right) + \partial_y\left(-\frac{\tau^x}{\rho f}\right) = curl\left(\frac{\vec{\tau}}{\rho f}\right) \approx \frac{1}{\rho f} curl\,\vec{\tau} \qquad (3.2.10)$$

In the subtropics, $w_e < 0$, and in the subpolar region $w_e > 0$, subsurface geostrophic currents will be generated.

<u>Example 2:</u> Atmospheric Spin-Down



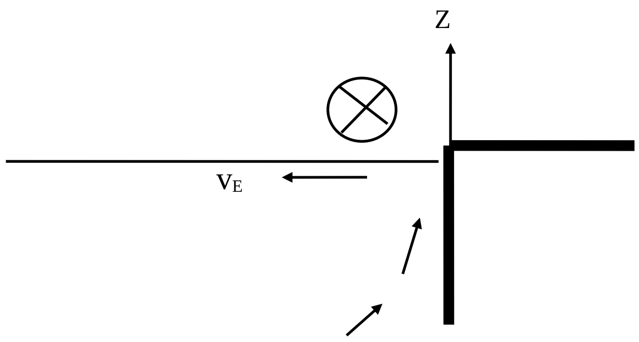
A cyclonic circulation (ζ>0) produces a convergent Ekman flow, which generates an Ekman upwelling $w_e > 0$. This compresses the water column above $H \downarrow$ and reduces the relative initial cyclonic vorticity $\zeta \downarrow$, and therefore acts as a spin down forcing due to the bottom friction. More specifically, since $\boldsymbol{\tau} \sim -\mathbf{u}_g$, $curl\boldsymbol{\tau} \approx -\nabla^2\psi$. Thus, on the RHS of the vorticity equation, the curl of the bottom stress appears as a damping

$$\partial_t q = curl\,\tau \approx -r\nabla^2\psi$$
$$\partial_t \nabla^2\psi = -r\nabla^2\psi$$

This indicates that $\frac{1}{r}$ is the spin down scale.

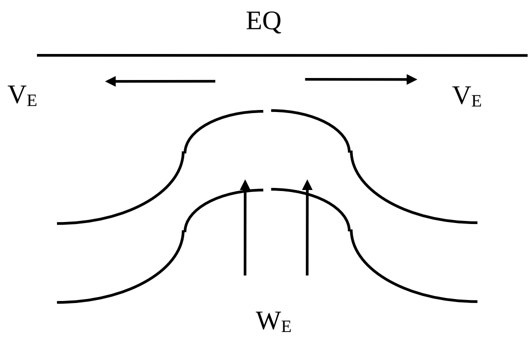
Example 3: Coastal Upwelling

Near the coastal region, even in the absence of Ekman pumping, an along shore wind can still generate coastal Ekman upwelling. First, the along shore wind (as shown below in the Southern Hemisphere) forces an off-shore Ekman flow. Mass compensation requires subsurface water to upwelling along the cast. This generates an Ekman upwelling. This process is important for coastal fishery and El Nino.



Example 4: Equatorial Upwelling

Equator also provides a natural boundary for the Ekman flow because f→0. So even under a uniform easterly wind, a divergent surface Ekman flow is produced, which is accompanied by an equatorial upwelling. This upwelling is of critical importance to El Nino process. Indeed, it is this upwelling that cools the SST on the equator. Consequently, the warmest surface water in the central-eastern tropical Pacific occurs off the equator, rather than right on the Equator (Fig.3.5), even through the solar radiation forcing is the strongest on the equator.



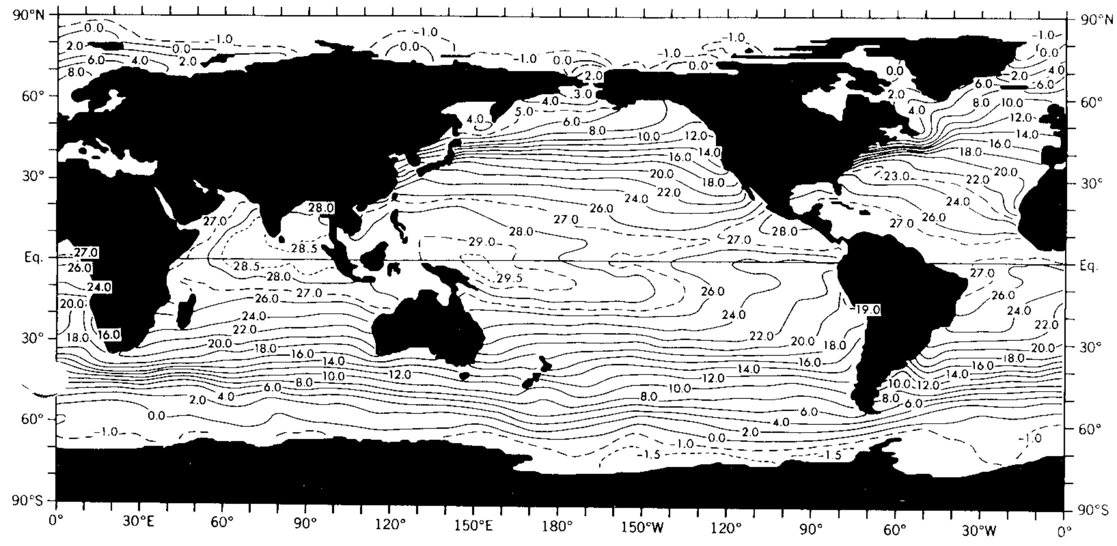


Fig.3.5 World Ocean annual mean SST

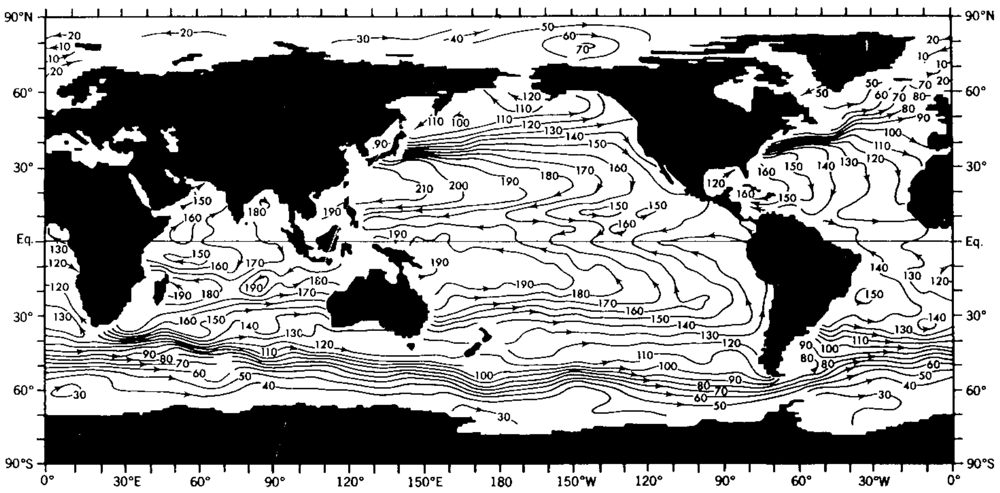
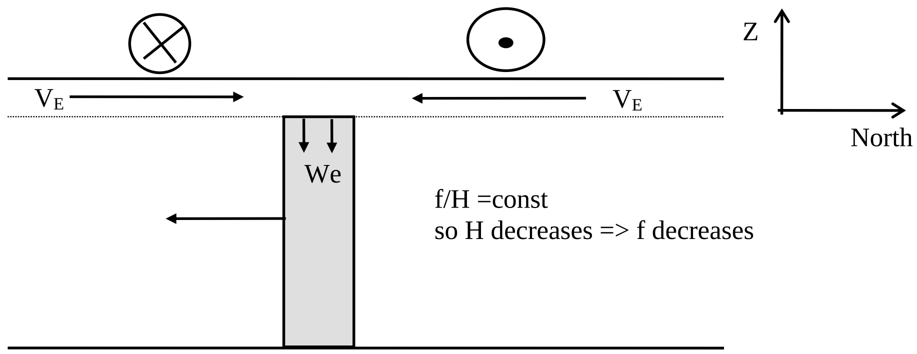


FIGURE 8.18. Global distribution of the annual-mean anomaly of the oceanic geopotential thickness in cm computed for the 0–1000 m layer (from Levitus and Oort, 1977).

Sec 3.3: Sverdrup Flow

1. The Sverdrup Flow

We consider how a downward Ekman Pumping, as in the subtropics, drives the ocean circulation.



Since we are considering large scales, relative vorticity is negligible. The potential vorticity is therefore $q = \dfrac{f+\xi}{H} \approx \dfrac{f}{H}$. Since a downward Ekman pumping $w_e < 0$ compresses the water column, H decreases. The conservation of potential vorticity requires that f also decreases, that is the water column must move southward.

This can be shown directly from the QGPV equation.

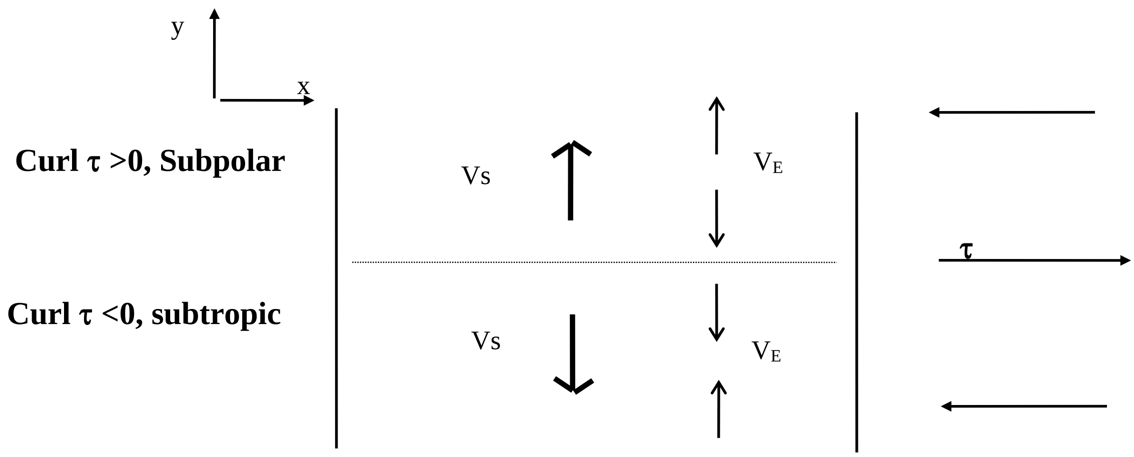
$$J(\psi, q) = \frac{curl\boldsymbol{\tau}}{\rho H} \approx \frac{f w_e}{H}$$

where $q = \nabla^2\psi + f_0 + \beta y \approx f_0 + \beta y$ because $\dfrac{\nabla^2\psi}{f_o + \beta y} \approx \varepsilon << 1$.

Thus, we have

$$\beta v_g H = f_0 w_e, \tag{3.3.1}$$

In the subtropical gyre, $w_e < 0$, the current is southward and in the subpolar gyre, $w_e > 0$ the current is northward. Eqn. (3.3.1) is the so called Sverdrup flow (in the context of the QG model).



N1: Sverdrup flow in the Planetary Geostrophic Model

More precisely, the Sverdrup relation can be derived directly from the PG equations which allows $\frac{\nabla f}{f} \approx O(1)$. In the PG equations, the subsurface geostrophic flow is governed by

$$\begin{cases} -fv_g = -g\eta_x \\ fu_g = -g\eta_y \\ u_{gx} + v_{gy} + w_z = 0 \end{cases} \tag{3.3.2}$$

Substitute the geostrophic flows from the first two equations into the continuity equation

$$\left(\frac{-g\eta_y}{f}\right)_x + \left(\frac{g\eta_x}{f}\right)_y + w_z = 0$$

$$\frac{-\beta}{f^2} g\eta_x + w_z = 0$$

$\beta v_g = f\partial_z w$

Integrate the water column beneath the Ekman layer $\int_{-H}^{\delta_z} dz$,

$$\beta V_g = f\left[w\left(-\delta_z\right) - w\left(-H\right)\right] = fw_e$$

where $V_g = \int_{-H}^{\delta_z} v_g dz$ is the total geostrophic transport of the water column. Therefore,

$$\boxed{\beta V_g = f w_e} \tag{3.3.3}$$

The transport of the total geostrophic flow beneath the Ekman Layer is determined by the Ekman pumping forcing.

The total flow is the sum of the geostrophic flow beneath the Ekman layer and the Ekman transport within the Ekman layer

$$V_s = \int_{-H}^{0} Vdz = \int_{-H}^{-\delta_z} Vdz + \int_{-\delta_z}^{0} Vdz = V_g + V_E = \frac{f}{\beta} curl\left(\frac{\tau}{\rho f}\right) - \frac{\tau^x}{\rho f} = \frac{1}{\beta} curl\left(\frac{\tau}{\rho}\right)$$

This is:

$$\boxed{\beta V_s = curl\left(\frac{\tau}{\rho}\right)} \tag{3.3.4}$$

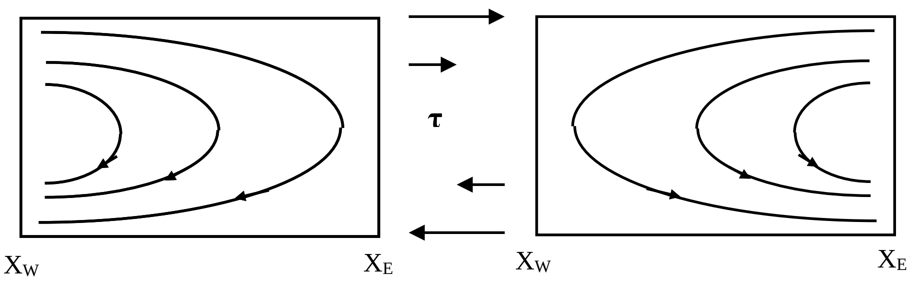
This is the true Sverdrup relationship. Notice that the Sverdrup flow consists of two parts, the surface Ekman flow and the subsurface geostrophic flow. The ratio of the two parts is:

$$V_E \big/ V_g \approx \frac{\frac{\tau}{\rho f}}{\frac{\tau}{\rho L \beta}} \approx \frac{\beta L}{f} \approx \frac{L}{a} << 1$$

in the mid-latitude. With the QG approximation, $fw_e = fcurl\left(\frac{\tau}{\rho f}\right) \approx curl\left(\frac{\tau}{\rho}\right)$ and the Sverdrup relation (3.3.4) degenerates to (3.3.1).

2. The problems with Sverdrup Flow

At the first sight, it seems that the ocean circulation problem is solved. This turns out to be not the case. In the following two figures, both flows are consistent with the Sverdrup flow (3.3.4) in the interior ocean. Yet, we could not decide which one is the correct one. Mathematically, this occurs because the Sverdrup flow (3.3.4) can't satisfy the no normal flow boundary condition on both the eastern and western boundaries simultaneously.



Satisfy the eastern B.C. Satisfy the western B.C.

Indeed if we solve the Sverdrup flow in the QGPV equation, we see this problem more clearly.

$$\beta\psi_x = curl\left(\frac{\vec{\tau}}{\rho H}\right) \tag{3.3.5}$$

A general solution to this equation is

$$\psi = \frac{1}{\beta\rho H}\int_{x'}^{x} curl(\vec{\tau})dx \quad \Rightarrow \begin{cases} v = \psi_x = \dfrac{1}{\beta\rho H} curl(\vec{\tau}) \\ u = -\psi_y = \dfrac{-1}{\beta\rho H} \int_{x'}^{x} \partial_y [curl(\vec{\tau})]dx \end{cases} \tag{3.3.6}$$

where *x'* is a fixed x position.

A proper solution should satisfy the no normal flow boundary condition on both the x_e and x_w:

$$\psi\big|_{X_e, X_w} = 0 \tag{3.3.7}$$

This assures the absence of a net meridional mass flux across the basin, as is required in the steady state solution. If we choose to satisfy the eastern boundary condition, we can set $x' = x_e$, so

$$\psi = \int_{x_e}^{x} \frac{curl(\tau)}{\beta\rho H} dx$$

To further satisfy the western boundary condition, we need

$$\psi = \int_{x_e}^{x_w} \frac{curl(\tau)}{\beta\rho H} dx = 0$$

This is too strong a constraint on the wind stress and is unlikely to hold in the general cases. For example, if $\boldsymbol{\tau} = \boldsymbol{\tau}(y)$, we have $\int_{x_e}^{x_w} curl(\tau)dx = curl(\tau)(x_e - x_w) \neq 0$. Therefore, one can satisfy either the eastern boundary condition with: $\psi = \int_{x_e}^{x} \frac{curl\boldsymbol{\tau}}{\beta\rho H} dx$ or the western boundary condition with $\psi = \int_{x_w}^{x} \frac{curl\boldsymbol{\tau}}{\beta\rho H} dx$, but not both!

Mathematically, the failure of the Sverdrup flow to satisfy both boundary conditions is obvious: the Sverdrup relation (3.3.6) has a first order derivative in x, so it can't satisfy the two meridional boundary conditions (3.3.7) at the same time. Selecting the boundary condition therefore requires higher order dynamics such as diffusion and nonlinearity, as will be discussed in the next section.

Physically, we can also understand why the ocean circulation solution is no longer a zonal flow. The meridional boundary blocks the PV contour such that, at the lowest order, the ocean circulation is not zonal flow (except in Antarctic Circumpolar Current, where no meridional boundary exists.).

Sec 3.4: Rossby Wave Adjustment, Ocean Circulation and Western Boundary Current

Here, we will examine how the steady Sverdrup flow is established, or the spin-up problem under a wind forcing. The general linear time-dependent QG equation can be written as:

$$\partial_t \, (\nabla^2 \psi - \psi/L_D^2) + \beta\partial_x \psi = curl\, \boldsymbol{\tau} - d\, \nabla^2 \psi \,, \tag{3.4.1}$$

where a bottom friction (or linear drag Rayleigh friction) has been used. After the sudden onset of a wind curl (for convenience, $\boldsymbol{\tau}=\boldsymbol{\tau}(y)$ is zonally uniform), the initial response is a forced zonal flow in the interior ocean governed by

$$\partial_t \, (\partial_{yy} \psi - \psi/L_D^2) = curl\, \boldsymbol{\tau} \qquad for\ t<X/C \tag{3.4.2}$$

where X is the distance of the point of consideration from the eastern boundary, and $C= -\beta L_D^2$ is the planetary wave speed. This forced zonal flow u= $-\partial_y \psi$ intensifies linearly with time, similar to the initial stage of a resonant response

$$\partial_{yy} \psi - \psi/L_D^2 = t\, curl\, \boldsymbol{\tau}.$$

However, this zonal flow satisfies neither the eastern nor the western boundary condition. Rossby waves are generated on the boundaries. The long waves, which satisfy

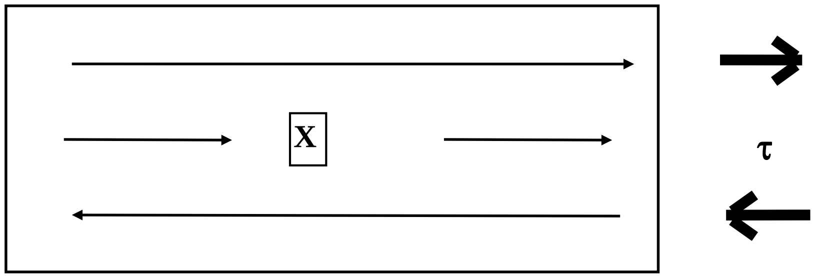
$$\partial_t \, (-\psi/L_D^2) + \beta\partial_x \psi = 0, \tag{3.4.3}$$

propagate westward from the eastern boundary at a group velocity C that is much faster than that of the short waves from the western boundary. After the passing of the long waves of the point of consideration, the flow reaches a steady state, as a forced planetary wave response:

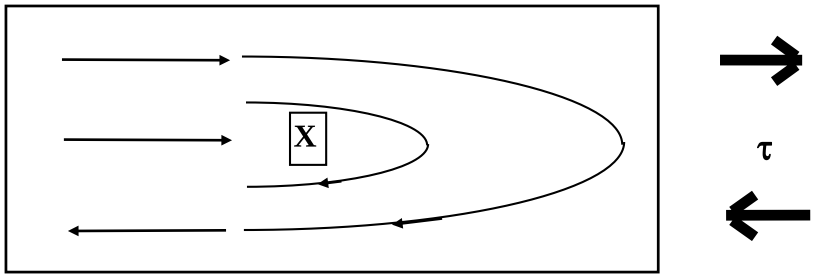
$$\beta\partial_x \psi = curl\, \boldsymbol{\tau}. \qquad for\ t>X/C \tag{3.4.4}$$

This is the Sverdrup flow, which is seen now as the superimposition of a forced and a free Rossby wave, as shown in (3.4.3) and (3.4.2), respectively. After the planetary wave across the basin width L ($t>L/C$), the Sverdrup flow is established in the entire basin.

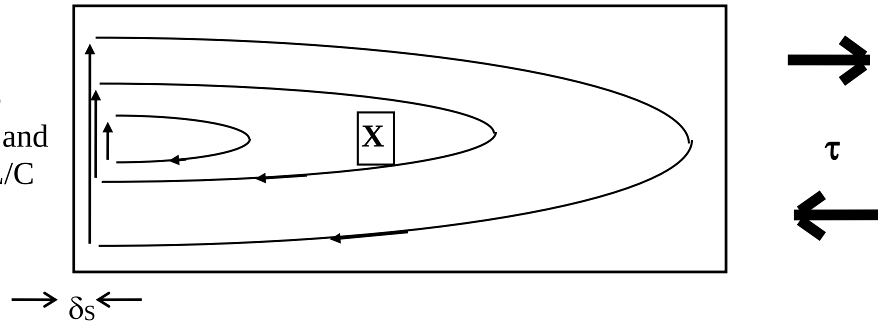
i) Initial response:
Forced zonal flow



ii) Planetary wave adjustment, establishing the local Sverdrup flow after $t>X/C$



iii) Basin-wide Sverdrup flow and WBC after t>L/C



The initial short Rossby waves are reinforced by the reflection of the long waves that cross the basin. All the short waves, however, have very slow eastward group velocity. Furthermore, these short waves are slow and of small scales, and therefore are subject to strong damping. Therefore, the short wave energy is trapped within the western boundary by friction and rapidly reaches the steady state:

$$\beta\partial_x \psi \ = \ -d\nabla^2\psi \ , \tag{3.4.5}$$

This produces the Stommel's western boundary layer, whose width can be derived from the scaling analysis as

$$\delta_S = d \,/\, \beta \, . \tag{3.4.6}$$

This boundary layer scale can also be seen from the Rossby wave viewpoint. The eastward group velocity of short Rossby waves is:

$$C_{gs} = \beta \,/\, k^2 .$$

The bottom friction time scale is:

$$T_B = 1/d.$$

The distance at which the waves are trapped by dissipation is therefore:

$$l_B = C_{gs}\, T_B = \beta / dk^2 \;.$$

Since the waves are short waves trapped within this friction distance, their wavelengths can't be much longer than l_B. Thus, we have $1/k \approx l_B$. Substitute it into the equation above for l_B, we recover the Stommel's boundary layer width $l_B = d/\beta = \delta_S$. The discussion above suggest that the beta effect and the friction are critical for the selection of the boundary layer in the west: the beta effect creates an east/west asymmetry associated with the Rossby wave, while the friction captures energy within a narrow boundary layer. The friction here is the higher order dynamics on the Sverdrup flow and enables us to establish the basin-wide steady circulation.

N1: We have seen another analogy of wave adjustment to an equilibrium state. We can compare the three types of adjustment in the following table. They share many similar features and in essence all represent the adjustment from one equilibrium to another.

Equilibrium State	Transient Waves	Adjustment Processes	Rotation Effect
Rest state, Flat surface	Gravity waves	Gravity wave adjustment	$f=0$
Geostrophic balance	Inertial-Gravity waves	Geostrophic adjustment	$f\neq 0$
Sverdrup flow and WBC	Rossby waves	Circulation spin-up	$\beta\neq 0$

N2: The effect of stratification.

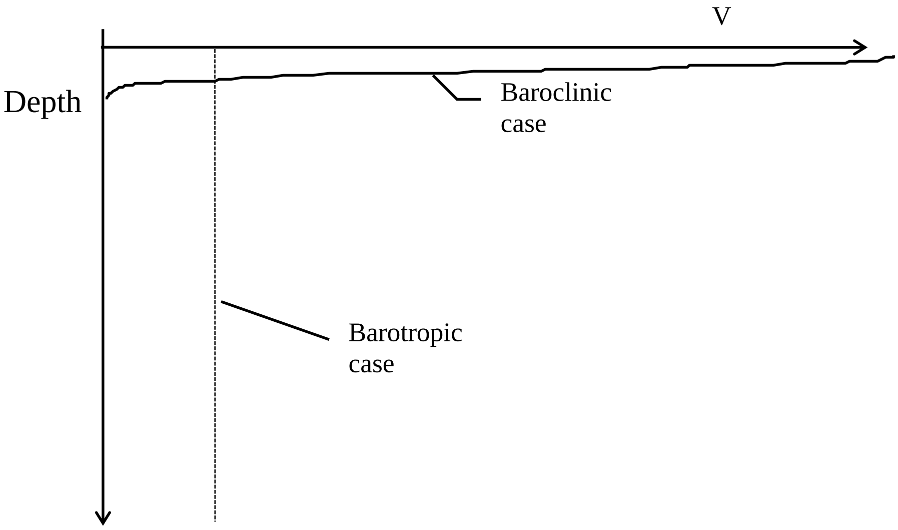
The Sverdrup flow is the vertically integrated transport and is independent of stratification (in the absence of bottom topography, as seen from the derivation of general Sverdrup relation (3.2.4) from the planetary geostrophic model. Therefore, the Sverdrup relationship remains valid in the stratified ocean, as long as there is no bottom topography. The Sverdrup relation, however, does not tell the vertical structure of the flow. In the case of a stratified ocean, we can show that there is no flow in the subsurface ocean at the final steady state. This follows because the linearized density equation becomes now:

$$w\partial_z \rho = 0$$

and therefore

$$w=0$$

if the stratification does not vanish $\partial_z\rho\neq 0$. With the meridional boundary, continuity equation further shows that $u=v=0$ beneath the Ekman layer. Therefore, the Sverdrup flow transport has to be trapped singularly beneath the Ekman layer as a delta function.


V
Depth
Baroclinic case
Barotropic case

Appendix: Rossby wave adjustment in a 1.5-layer QG model:

$$\partial_t q+\beta\partial_x\psi=w_e(t)$$

where the PV is

$$q=\partial_{xx}\psi+\partial_{yy}\psi-\psi/L^2_D.$$

The equation is nondimensionalized such that $\beta=1,\ L^2_D=1,$ The variables are nondimensionlized with ψ by βL^2_D, x by L_D and t by $1/\ \beta L^2_D$,.

Case 1: Initial value problem: initial short wave case

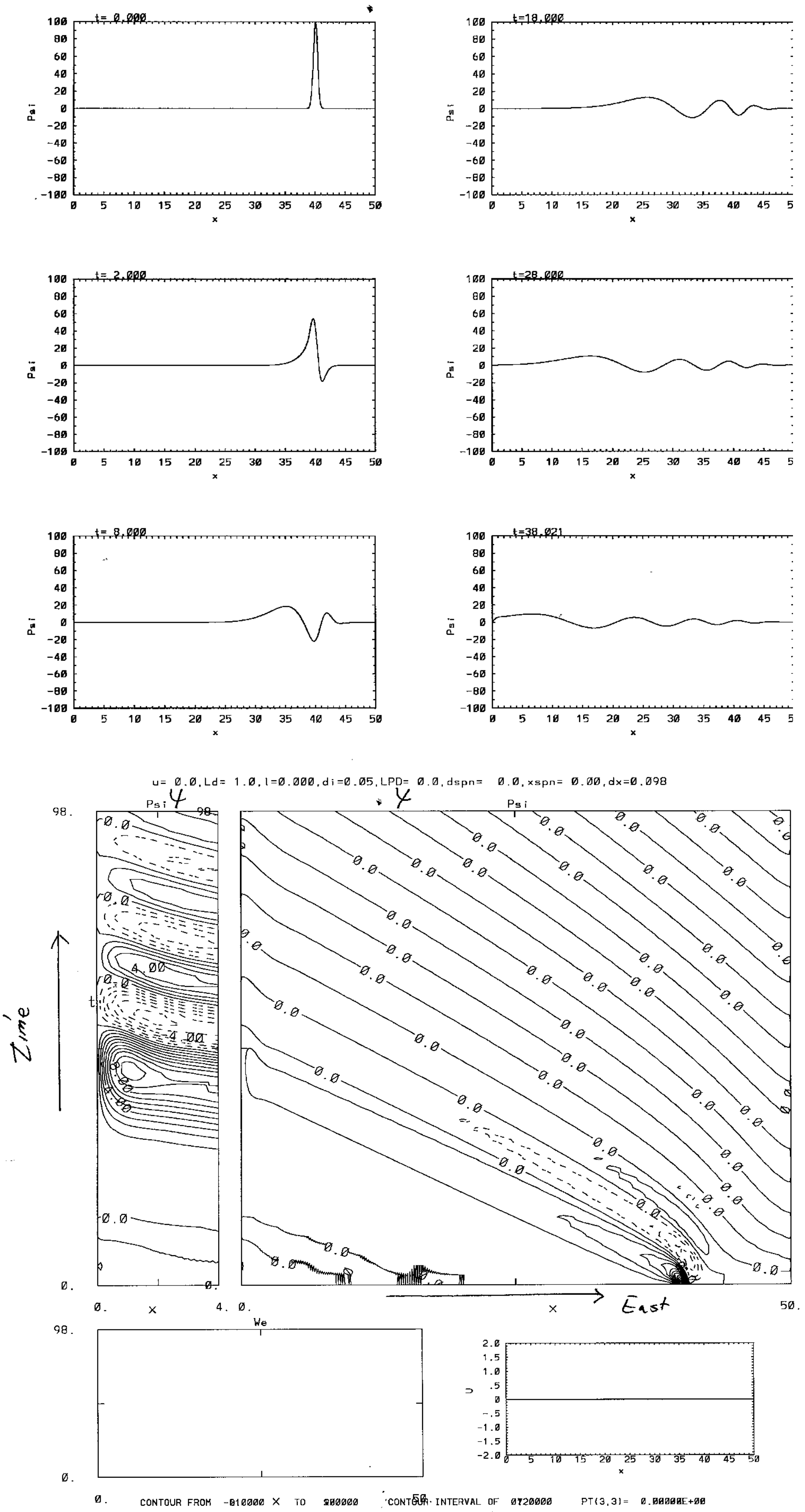
Case 2: Initial value problem: initial long wave case

Case 3: forced wave radiation, fast forcing ($\sigma=2$),

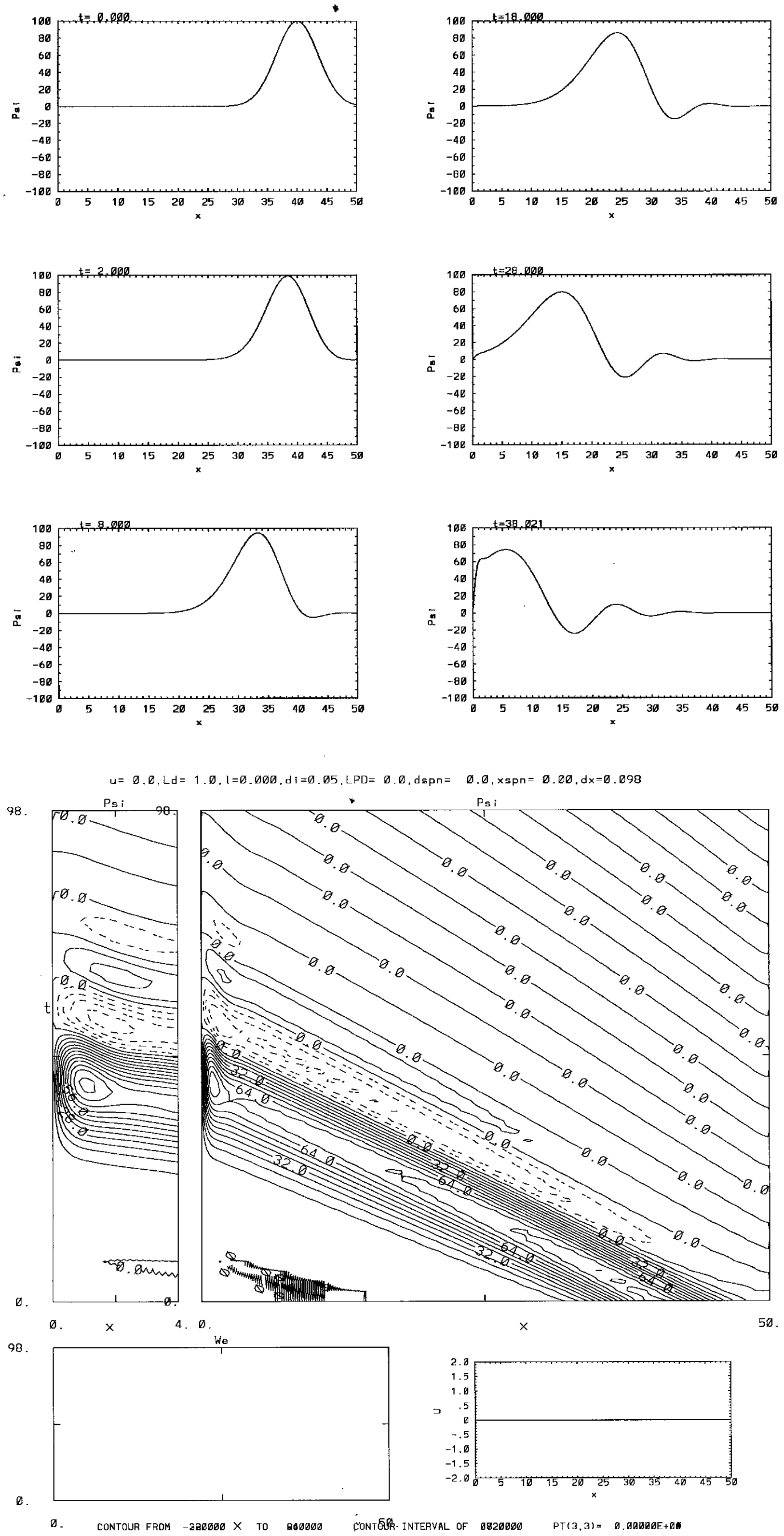
Case 4: forced wave radiation, slow forcing ($\sigma=0.5$),

Case 5: forced wave radiation, ultra-slow forcing ($\sigma=0.1$),

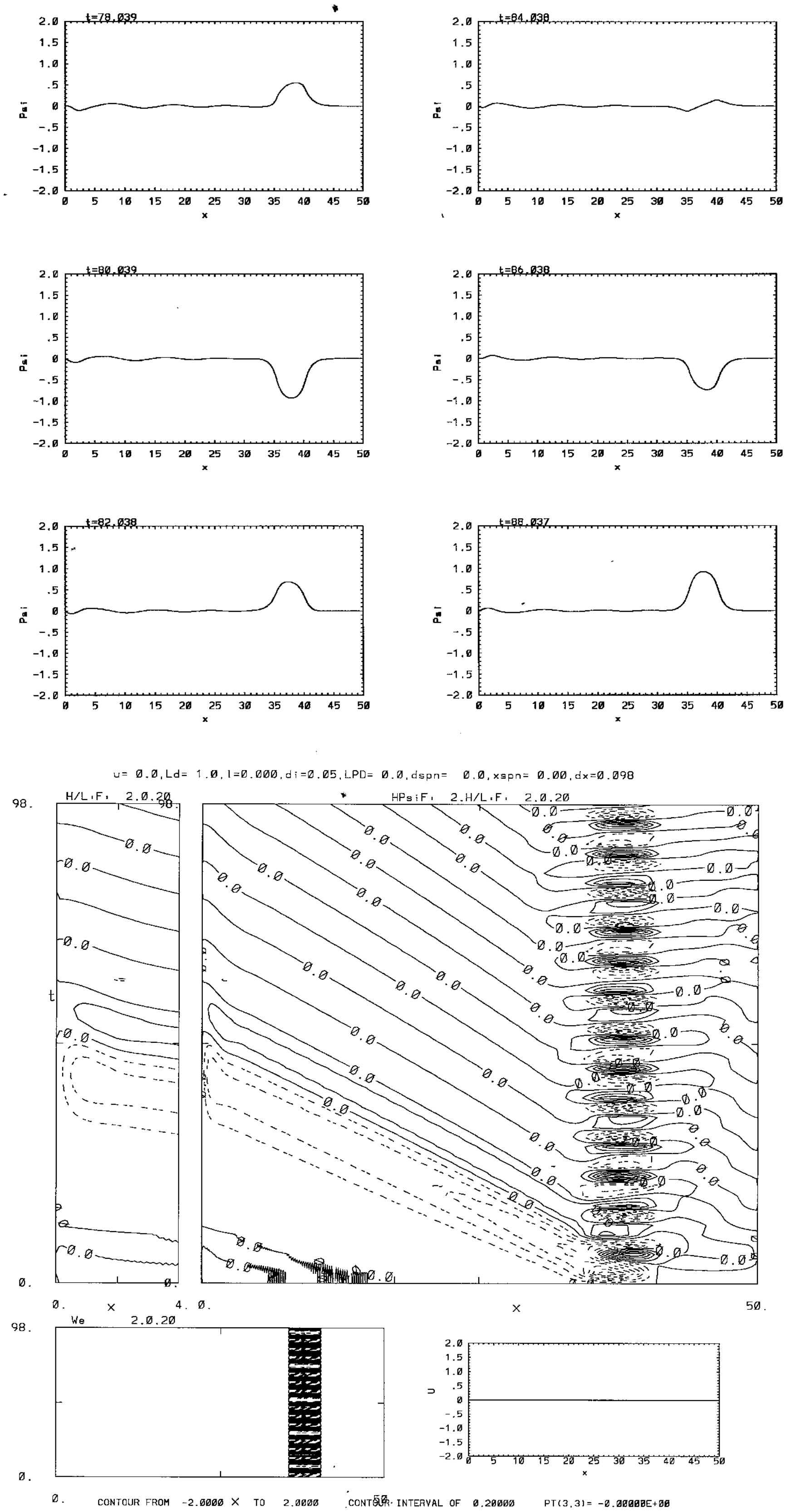
Case 6: spin-up forcing, Sverdrup flow and the western boundary current


t= 0.000
t= 2.000
t= 8.000
t=10.000
t=20.000
t=30.021
Psi
x
u= 0.0,Ld= 1.0,l=0.000,di=0.05,LPD= 0.0,dspn= 0.0,xspn= 0.00,dx=0.098
Time
East
We
CONTOUR FROM -010000 TO 200000 CONTOUR INTERVAL OF 0T20000 PT(3,3)= 0.00000E+00

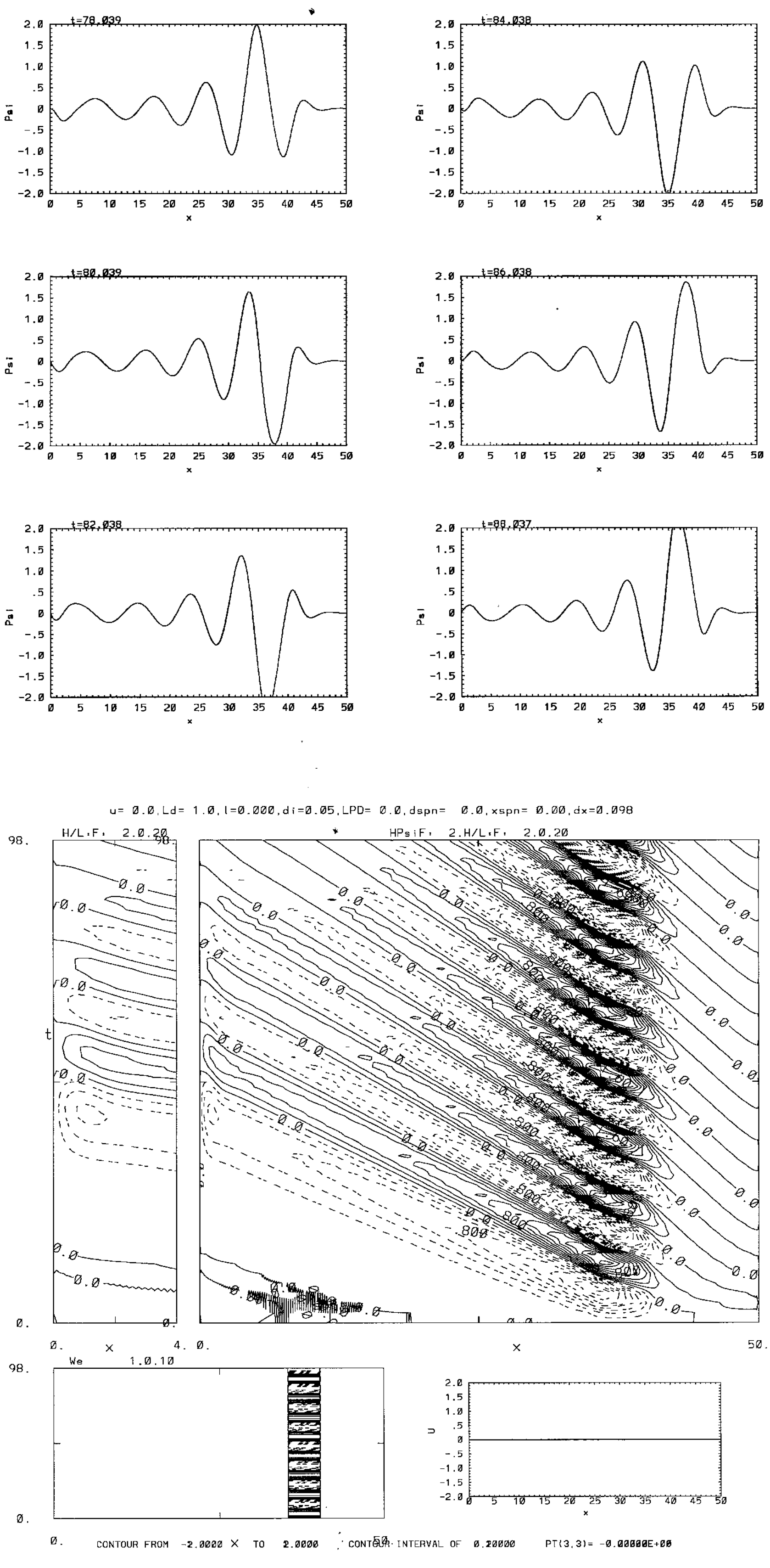
Case 1


t= 0.000
t=18.000
t= 2.000
t=26.000
t= 8.000
t=39.021
Psi
x
u= 0.0,Ld= 1.0,l=0.000,dl=0.05,LPD= 0.0,dspn= 0.0,xspn= 0.00,dx=0.098
We
CONTOUR FROM -320000 X TO 960000 CONTOUR INTERVAL OF 0320000 PT(3,3)= 0.00000E+00

Case 2

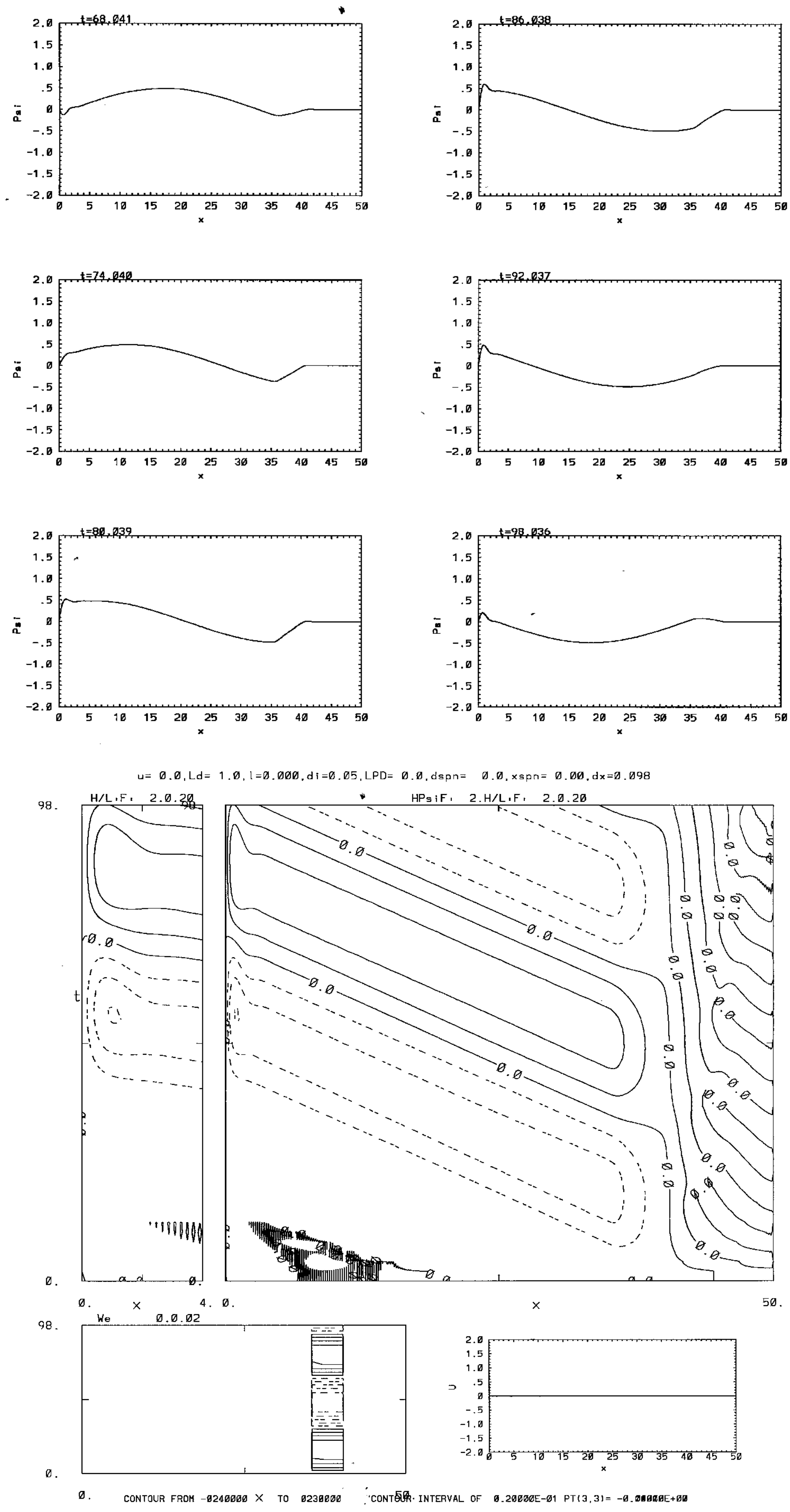


Case 3

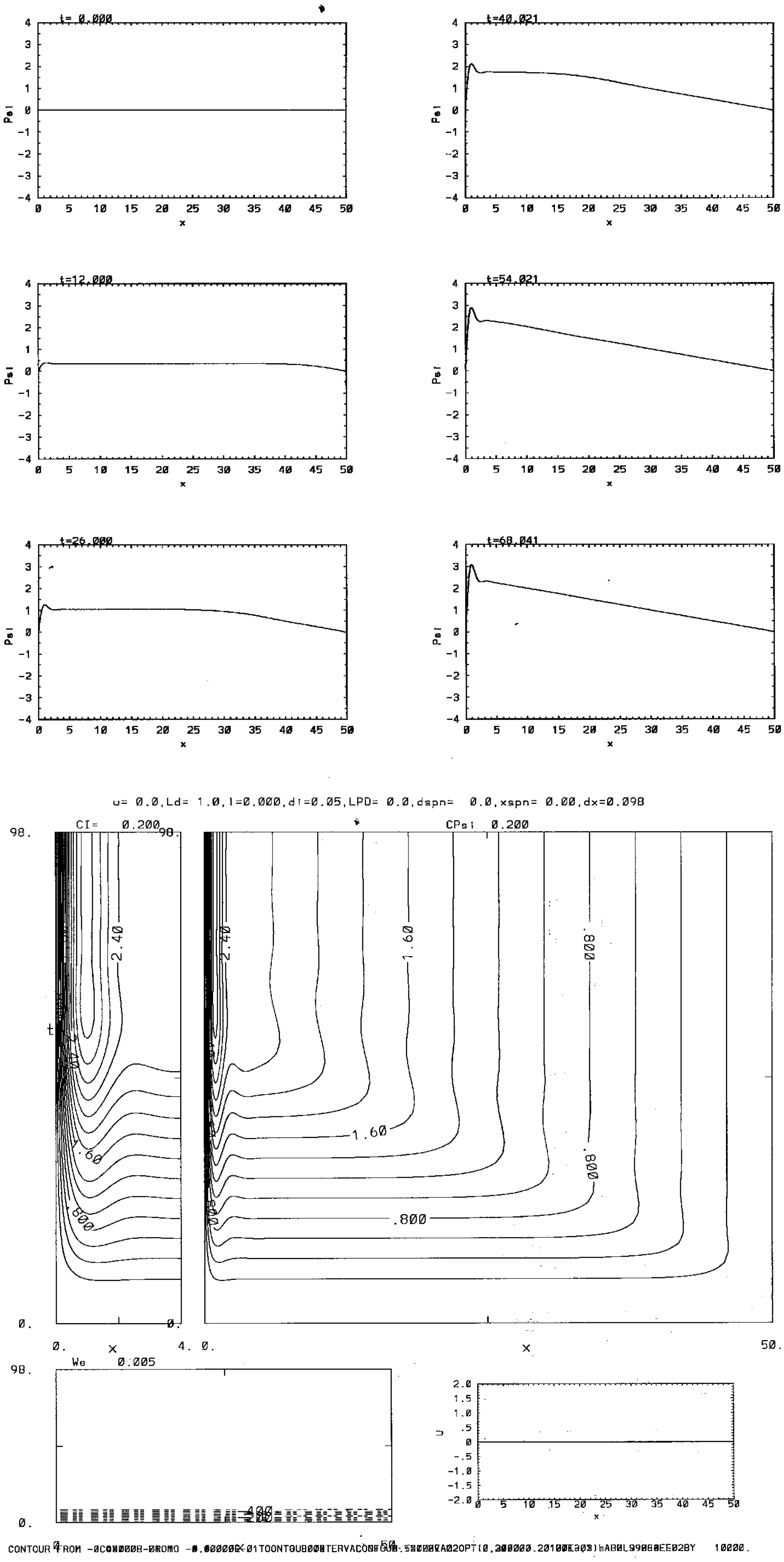


t=78.039
t=84.038
t=80.039
t=86.038
t=82.038
t=88.037
Psi
x
u= 0.0,Ld= 1.0,l=0.000,di=0.05,LPD= 0.0,dspn= 0.0,xspn= 0.00,dx=0.098
H/L,F, 2.0.20
HPsiF, 2.H/L,F, 2.0.20
t
We 1.0.10
CONTOUR FROM -2.0000 TO 2.0000 CONTOUR INTERVAL OF 0.20000 PT(3,3)= -0.00000E+00

Case 4


t=68.041
t=86.038
t=74.040
t=92.037
t=80.039
t=98.036
Psi
x
u= 0.0,Ld= 1.0,l=0.000,dl=0.05,LPD= 0.0,dspn= 0.0,xspn= 0.00,dx=0.098
H/L,F, 2.0.20
HPsiF, 2.H/L,F, 2.0.20
We 0.0.02
CONTOUR FROM -0240000 X TO 0230000 CONTOUR INTERVAL OF 0.20000E-01 PT(3,3)= -0.00000E+00

Case 5


t= 0.000
t=12.000
t=26.000
t=40.021
t=54.021
t=68.041
Psi
x
u= 0.0,Ld= 1.0,l=0.000,dl=0.05,LPD= 0.0,dspn= 0.0,xspn= 0.00,dx=0.098
CI= 0.200
CPsi 0.200
2.40
1.60
.800
We 0.005
u

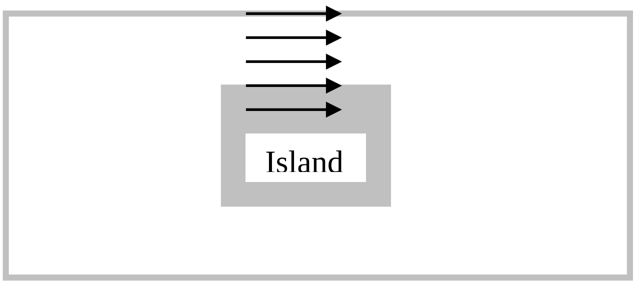
Case 6

Questions for Chapter 3

Q3.1. What happens to the western boundary layer when beta is reduced? Does the western boundary layer still exist when beta vanishes? Why?

Q3.2. Can the atmosphere have western boundary intensification and western boundary current? Where do you think it most likely to occur?

Q3.3. An island exists in a northern hemisphere ocean basin. Suppose a uniform westerly wind stress is switched on at t=0, how would the ocean response? Would there be a circulation at its final state?



Exercises for Chapter 3

E3.1. (Ekman Spiral) Suppose the Ekman layer is governed by $-fv=A\partial_{zz}u$, $fu=A\partial_{zz}v$, where the viscosity A is a constant. For a given wind stress in y direction, $A\partial_z v|_{z=0}=\tau/\rho$, verify that the Ekman layer solution satisfies

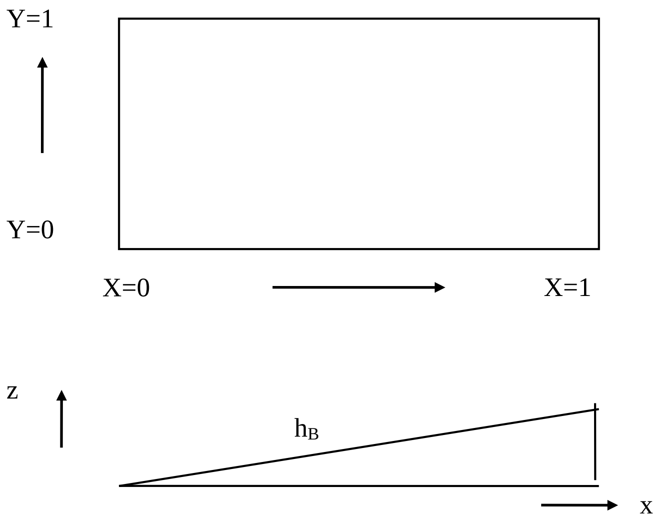
$$u=V_0\cos[\pi/4+z/D_E]\exp(z/D_E),\quad v=V_0\sin[\pi/4+z/D_E]\exp(z/D_E).$$

where $V_0=2^{1/2}\tau/(D_E f\rho)$, and $D_E=(2A/f)^{1/2}$. Discuss the structure of the Ekman spiral.

E3.2. (Ocean circulation in the presence of topography) A homogeneous ocean in a Northern Hemisphere ocean basin is forced by a negative wind curl. The ocean floor has a slope that shallows towards the east. Suppose the linear QG dynamics applies and the friction is bottom friction (or Raleigh friction) such that the flow is governed by

$$J(\psi,\ \beta y+h)=curl\ \boldsymbol{\tau}-r\nabla^2\psi$$

where $h=f_0h_B/H$ and $h_B=ax$ is the bottom topography with $a>0$. The wind curl is negative within the basin $curl\ \boldsymbol{\tau}<0$ everywhere. (a) Find the interior ocean circulation, (b) draw schematically the basin circulation and describe the location of the boundary layers.



E3.3. (Advective-diffusive boundary layer) In a 1-D pipe $0\le x\le 1$, the temperature of the fluid is governed by the steady advective-diffusive equation $u\partial_x T=A\,\partial_{xx}T$, where u is a constant velocity and $A>0$ is a constant diffusivity. The boundary conditions are: $T(x=0)=0$, and $T(x=1)=1$.

(a) Find the exact solution analytically, for the three cases: $u>0$, $u=0$ and $u<0$.

(b) Under what conditions, there exists a boundary layer? Where is it?

(c) Do you see a similarity of the above result with the Stommel's boundary layer in a wind-driven gyre.

E3.4: (Wind forced oceanic response) An open upper ocean is governed by the 1.5-layer QG model $\partial_t(-\frac{\psi}{L_D^2})+\beta\partial_x\psi=\frac{f_0 w_E}{H}$, where the Ekman pumping forcing has the form $w_E(t)=w_0 \cos(\sigma t)$. (a) What is the forced oceanic response if the ocean is unbounded in x? What happens when the forcing frequency σ approaches zero? (b) What is the forced oceanic response if the ocean is bounded by an eastern boundary at $x=0$? What happens when the forcing frequency σ approaches zero? (c) In the solution of (b), if we only consider the oceanic response east of x_w (<0), what happens when the forcing frequency σ approaches zero?

E3.5: (Wind forced oceanic response) Repeat ***E3.4,*** but keep the relative vorticity such that the equation is $\partial_t(\nabla^2\psi-\frac{\psi}{L_D^2})+\beta\partial_x\psi=\frac{f_0 w_E}{H}$.

E3.6: (Planetary wave basin mode) In a rectangular basin $0\le x\le X$, $0\le y\le Y$, find the baroclinic planetary wave basin modes that satisfy the eastern boundary condition and basin wide mass conservation condition. In the QG context, the planetary wave equation is $\partial_t\psi-C\partial_x\psi=0$, where $C=\beta L_D^2=\beta g'D/f_0^2$; the eastern boundary condition of no normal flow is equivalent to $\psi|_{x=X}=\psi_o(t)$ (NOT necessarily 0!) and the basin wide mass conservation becomes $\partial_t\iint_{Basin}\psi dxdy=0.$ The basin normal mode can be assumed of the form $\psi=e^{\omega t}G(x,y)$. (a) To satisfy the equation and the eastern boundary condition, show that the solution has the form $\psi=Ae^{\omega(t+\frac{x}{C})}$, where A is a constant. (b) To satisfy the mass conservation condition, show that the eigenvalues are $\omega=2n\pi i/T$ $(n=1, 2, 3\ldots.)$ where $T=X/C$ is the transient time of the Rossby wave across the basin (c) Plot the eigenfunction of mode 1 and 2 at the phases $\omega t=0$, $i\pi/2$, $i\pi$ and $i3\pi/2$. (d) How does the eastern boundary change with time? (e) Interpret the formation process of the planetary wave normal mode, in light of the basin adjustment study of Liu et al., 1999 (JPO, 29, p2383-p2404). (f) If the planetary wave speed varies with latitude (that is $C=\beta g'D/f^2$ and $f=f_0+\beta y$), would the basin mode still be neutral modes?