

## Ch.5: Stratified Quasi-Geostrophic Rossby Waves

### Sec. 5.1: Quasi-Geostrophic Equation in Stratified Fluid

#### 1. Nondimensional Equations

We'll use the oceanic equations (4.1.15)

$$\begin{aligned}\partial_t u + (\mathbf{u} \cdot \nabla)u - fv &= -\frac{1}{\rho_0} \partial_x p' + \frac{1}{\rho_0} F_x \\ \partial_t v + (\mathbf{u} \cdot \nabla)v + fu &= \frac{1}{\rho_0} \partial_y p' + \frac{1}{\rho_0} F_y \\ \frac{1}{\rho_0} \partial_z p' &= -\frac{g\rho'}{\rho_0} \\ \partial_x u + \partial_y v + \partial_z w &= 0 \\ \partial_t \rho' + (\bar{\mathbf{u}} \cdot \nabla)\rho' &= S_0\end{aligned}\tag{5.1.1}$$

We choose the scales as

$$u, v \sim U, \quad x, y \sim L, \quad w \sim U \frac{D}{L}, \quad z \sim D, \quad t \sim \frac{L}{U}$$

and denote the Rossby number as:

$$\varepsilon = \frac{U}{f_o L}$$

The density and pressure can be written as

$$\rho'(x, y, z, t) (= \rho_{Ocean} - \rho_0) = \rho_s(z) + \rho(x, y, z, t),\tag{5.1.2a}$$

$$P'(x, y, z, t) (= P_{Ocean} - \rho_0 g z) = P(z) + p(x, y, z, t)\tag{5.1.2b}$$

where  $\rho_{Ocean}$  and  $P_{Ocean}$  are the total density and pressure,  $\rho_0$  is the average density of the ocean, and

$$\frac{dP}{dz} = -g\rho_s(z)$$

represents the static part associated with the mean stratification and

$$\frac{\partial p}{\partial z} = -g\rho$$

is the dynamic pressure associated with the horizontal density variations. For large scale flows with small Rossby number, similar to the shallow water case in section 2.1, the dynamic pressure is also scaled as

$$p \sim f_o L U \rho_0\tag{5.1.3}$$

such that the pressure gradient force is comparable with the Coriolis force; the source/sink is assumed weak, with  $\frac{1}{f_o U \rho_0} \mathbf{F} = \varepsilon \mathbf{G}$ , and  $G \leq O(1)$ ; we also use the local  $\beta$ -plane approximation

$\frac{\beta L}{f_o} = \gamma \varepsilon$ , where  $r \sim 1$ . Then, we can write the two momentum equations in dimensionless

variables (subscripted with “\*”) as:

$$\varepsilon \left\{ \frac{\partial u_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) u_* - \gamma y_* v_* \right\} - v_* = -\frac{\partial \hat{p}_*}{\partial x_*} + \varepsilon G_x$$

$$\varepsilon \left\{ \frac{\partial v_*}{\partial y_*} + (\mathbf{u}_* \cdot \nabla_*) v_* + \gamma y_* u_* \right\} + u_* = -\frac{\partial \hat{p}_*}{\partial y_*} + \varepsilon G_y$$

The continuity equation is

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{\partial w_*}{\partial z_*} = 0$$

The scale of  $\rho$  can be derived from the hydrostatic equation

$$\frac{\partial \hat{p}}{\partial z} = -g\rho.$$

Notice (5.1.3), we have

$$\rho \sim \frac{p}{gD} \sim \frac{f_o LU}{gD} \rho_0 \sim \rho_0 \left( \frac{L}{L_D} \right)^2 \varepsilon \equiv \Gamma \quad (5.1.4)$$

where  $L_D^2 = \frac{gD}{f_o^2}$  is the *external* deformation radius. We can therefore define  $\rho$  as

$$\rho \sim \Gamma \rho_* \quad (5.1.4a)$$

where  $\rho_* \sim O(1)$ . The hydrostatic equation can be written as

$$\frac{\partial \hat{p}_*}{\partial z_*} = -\rho_*$$

With all these scalings, the thermodynamic equation becomes

$$\frac{U\Gamma}{L} \left\{ \frac{\partial \rho_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) \rho_* \right\} + \frac{UD}{L} w_* \frac{d\rho_s}{dz} = S_0.$$

Thus,

$$\varepsilon \left\{ \frac{\partial \rho_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) \rho_* \right\} + \frac{UD}{f_o L \Gamma} w_* \frac{d\rho_s}{dz} = \frac{S_0}{f_o \Gamma}$$

$$\varepsilon \left\{ \frac{\partial \rho_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) \rho_* \right\} + \frac{D\varepsilon}{\Gamma} w_* \frac{d\rho_s}{dz} = \frac{S_0}{f_o \Gamma} \quad (5.1.5)$$

The scale of  $\frac{d\rho_s}{dz}$  can be derived at the first order from the adiabatic condition. We can show from (5.1.5) that

$$\frac{d\rho_s}{dz} \geq \frac{1}{\varepsilon} \frac{\partial \rho}{\partial z} \gg \frac{\partial \rho}{\partial z}$$

Indeed, for adiabatic flows, the solution is always along isopycnals  $\frac{d\rho'}{dt} = 0$ . Therefore,  $u \partial_x \rho' \approx w \partial_z \rho'$ . But, we know that QG equations require at the first order non-divergent:

$$\frac{w}{u} \leq \varepsilon \frac{D}{L}$$

Therefore, the slope of the isopycnal surface must be  $\leq \varepsilon \frac{D}{L}$

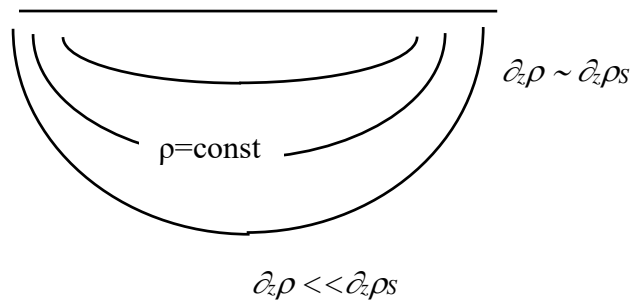
$$\frac{\partial \rho'}{\partial x} \bigg/ \frac{\partial \rho'}{\partial z} \approx \frac{w}{u} \leq \varepsilon \frac{D}{L}$$

Since  $\frac{\partial \rho_s}{\partial x} = 0$ , we have  $\frac{\partial \rho'}{\partial x} = \frac{\partial \rho}{\partial x} \sim \frac{\partial \rho}{\partial z} \frac{D}{L}$ , and therefore

$$\frac{\partial \rho}{\partial z} \bigg/ \frac{\partial(\rho_s + \rho)}{\partial z} \leq \varepsilon$$

$$\frac{\partial \rho}{\partial z} \bigg/ \frac{d\rho_s}{dz} \leq \varepsilon \tag{5.1.6}$$

This implies that any horizontal variation of the static stability (since  $\rho'$  is the horizontal variation part) must be small. This is a weak assumption in many cases. (especially in marine eddies.)



Using (5.1.4) and (5.1.6), we have the scale

$$\frac{d\rho_s}{dz} = \frac{\Gamma}{D\varepsilon} \Gamma_*(z)$$

where  $\Gamma_*(z) \leq O(1)$ .

The thermodynamic equation becomes

$$\varepsilon \left\{ \frac{\partial \rho_*}{\partial t_*} + (\mathbf{u}_* \cdot \nabla_*) \rho_* \right\} + w_* \Gamma_* = \varepsilon S$$

where  $\varepsilon S = \frac{S_0}{f_o \Gamma}$  and  $S \leq O(1)$  such that at the leading order the flow is adiabatic.

Note 1: If we define a buoyancy frequency  $N(z)$

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_s}{dz}$$

we have

$$O(1) \geq \Gamma_*(z) = \frac{\frac{d\rho_s}{dz}}{\Gamma/D\varepsilon} = \frac{N^2 \rho_0 / g}{(\rho_0 L^2 \varepsilon f_0^2 / gD) / D\varepsilon} = \left(\frac{L_I}{L}\right)^2$$

where  $L_I^2 = (ND/f)^2$  is the *interior* deformation radius.

Therefore,

$$\frac{\partial \rho}{\partial z} \leq \varepsilon \frac{d\rho_s}{dz} \Leftrightarrow \left(\frac{L}{L_I}\right)^2 \leq 1$$

This requires the scale to be not too large, similar to the homogeneous case.

The complete set of dimensionless equations is therefore (drop the subscript “\*\*”):

$$\begin{aligned} v - \frac{\partial \hat{p}}{\partial x} &= \varepsilon \left\{ \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) u - ryv - G_x \right\} \\ -u - \frac{\partial \hat{p}}{\partial y} &= \varepsilon \left\{ \frac{\partial v}{\partial t} + (\mathbf{u} \cdot \nabla) v + ryu - G_y \right\} \\ -w\Gamma_* &= \varepsilon \left\{ \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho - S \right\} \end{aligned} \tag{5.1.7}$$

$$\frac{\partial \hat{p}}{\partial z} = -\rho$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Similar to the shallow water case in Section 2.1, we solve this set of equations by expanding variables as powers of  $\varepsilon$ :

$$\begin{aligned} u &= u_o + \varepsilon u_1 + O(\varepsilon^2) \\ v &= v_o + \varepsilon v_1 + O(\varepsilon^2) \\ p &= p_o + \varepsilon p_1 + O(\varepsilon^2) . \\ \rho &= \rho_o + \varepsilon \rho_1 + O(\varepsilon^2) \\ w &= w_o + \varepsilon w_1 + O(\varepsilon^2) \end{aligned} \tag{5.1.8}$$

## 2. O(1) Equation and Dynamics

Substitute (5.1.8) into (5.1.7), at the leading order, we have

$$\begin{aligned}
 v_o &= \frac{\partial \hat{p}_o}{\partial x} \\
 u_o &= -\frac{\partial \hat{p}_o}{\partial y} \\
 w_o &= 0 \\
 \frac{\partial \hat{p}_o}{\partial z} &= -\rho_o \\
 \frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} + \frac{\partial w_o}{\partial z} &= 0
 \end{aligned} \tag{5.1.9a-e}$$

As in the shallow water case, (a), (b), (c) can be used to derive (e). Thus, there are only 4 independent equations, but with 5 unknowns. This is the ‘‘Geostrophy degeneracy’’.

To better understand the O(1) dynamics, we write (5.1.9a,b) in dimensional form as the geostrophic balance:

$$\begin{aligned}
 v_g &= \frac{1}{f_o \rho_o} \frac{\partial \hat{p}}{\partial x} \equiv \partial_x \psi \\
 u_g &= -\frac{1}{f_o \rho_o} \frac{\partial \hat{p}}{\partial y} \equiv -\partial_y \psi
 \end{aligned} \tag{5.1.10}$$

where  $\psi = \frac{p}{\rho_o f_o}$  is the geostrophic stream function. The hydrostatic balance (5.1.9c) can be written as

$$\frac{\rho}{\rho_o} = -\frac{f_o}{g} \frac{\partial \psi}{\partial z} \tag{5.1.11}$$

Differentiate (5.1.10) with respect to  $z$  and use (5.1.11), we have

$$\begin{aligned}
 \frac{\partial v_g}{\partial z} &= \frac{\partial^2 \psi}{\partial x \partial z} = -\frac{g}{f_o \rho_o} \frac{\partial \rho}{\partial x} \\
 \frac{\partial u_g}{\partial z} &= -\frac{\partial^2 \psi}{\partial y \partial z} = \frac{g}{f_o \rho_o} \frac{\partial \rho}{\partial y}
 \end{aligned} \tag{5.1.12}$$

This is the thermal wind relation, a direct result of geostrophy and hydrostatic balance. This relation has been used frequently to infer ocean currents from the density field, i.e. the so called ‘‘dynamic method’’.

Note 2: It also represents a balance between the baroclinic vorticity generation and the title of planetary vorticity in  $y$ - and  $x$ - directions. Indeed, the general vorticity equation is:

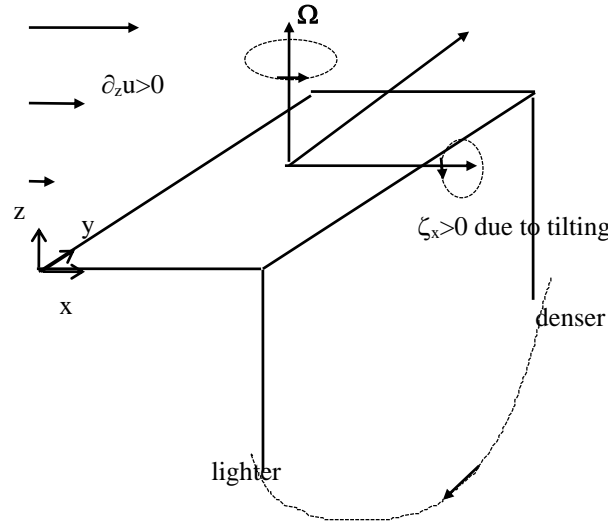
$$\frac{d\xi_a}{dt} = (\xi_a \bullet \nabla) \mathbf{u} - \xi_a \nabla \bullet \mathbf{u} + \frac{\nabla \rho \times \nabla p}{\rho^2}$$

For large scale  $\varepsilon \ll 1$ , we have  $\xi_a = 2\Omega + O(\varepsilon)$ . In addition,  $\partial_z \rho \partial_y P \ll \partial_y \rho \partial_z P$ , which is equivalent to the Boussinesq approximation in the ocean. The x and y component of the vorticity equation can therefore be written as

$$\frac{d\xi_{ax}}{dt} \approx 2\Omega \partial_z u + \frac{\partial_y \rho \partial_z P - \partial_z \rho \partial_y P}{\rho^2} \approx 2\Omega \partial_z u + \frac{\partial_y \rho \partial_z P}{\rho^2}$$

$$\frac{d\xi_{ay}}{dt} \approx 2\Omega \partial_z v + \frac{\partial_z \rho \partial_x P - \partial_x \rho \partial_z P}{\rho^2} \approx 2\Omega \partial_z v - \frac{\partial_x \rho \partial_z P}{\rho^2}$$

Notice hydrostatic balance:  $\partial_z p = -\rho g$ , in the steady state  $\frac{d\xi_{ax}}{dt} = \frac{d\xi_{ay}}{dt} = 0$ , we have the thermal wind relationship. On the RHS in the two equations above, the first term is the tilting term while the second term is the baroclinic term. The balance can be seen schematically as follows:



A westerly wind shear  $\partial_z u > 0$  generates positive vorticity in the x direction  $\Omega \partial_z u > 0 \Rightarrow \partial_x \xi > 0$ . This vorticity is balanced by the opposite rotation that is forced by the northward density gradient  $\partial_y \rho > 0, \partial_z p < 0 \Rightarrow \partial_y \rho \partial_z p < 0$ .

Note 3: Taylor - Proudman Theorem

For large scale low frequency processes, we have  $\xi_a \approx 2\Omega$ , and  $\partial_t < \Omega$ . The vorticity equation, assuming incompressibility, is

$$(2\Omega \bullet \nabla) \mathbf{u} = \frac{\nabla \rho \times \nabla p}{\rho^2}$$

If furthermore, the fluid is barotropic  $\nabla \rho \times \nabla p = 0$ , we have

$$(\Omega \bullet \nabla) \mathbf{u} = 0$$

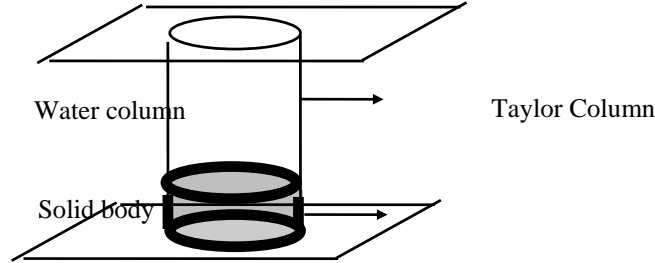
or assuming  $\Omega = \Omega \mathbf{k}$ , this is

$$\Omega \partial_z \mathbf{u} = 0$$

or there is no shear of velocity in the vertical direction.

$$\partial_z u = \partial_z v = \partial_z w = 0$$

The water column therefore behaves like a column of solid body.



Even in stratified case, this still shows the tendency of rotation to constraint fluid motion variation along  $\Omega$ . In other words, rotation tends to couple the flow in the direction of  $\Omega$ . Thus, rotation produces a “stiffening” effect that tends to align the vortex tube in the direction of rotation. This is also why in a layered model we can assume no shear for large scales GFD processes.

### 3. $O(\varepsilon)$ Equation and QGPV Equation

At the next order, we have the equations

$$v_1 - \frac{\partial \hat{p}_1}{\partial x} = \frac{D_{*g}}{Dt} u_o - ryv_o - G_x$$

$$-u_1 - \frac{\partial \hat{p}_1}{\partial y} = \frac{D_{*g}}{Dt} v_o + ryu_o - G_y$$

$$-w_1 \Gamma_* = \frac{D_{*g}}{Dt} \rho_o - S \quad (5.1.13)$$

$$\frac{\partial \hat{p}_1}{\partial z} + \rho_1 = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0$$

where  $\frac{D_{*g}}{Dt} = \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}$ . (This becomes a horizontal total derivative because  $w_o = 0$ ). The

vorticity equation is therefore:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = -\frac{D_{*g}}{dt} \left( \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} \right) - rv_o + \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}$$

Here, we used:  $\frac{\partial}{\partial x} \left( \frac{D_{*g}}{dt} v_o \right) - \frac{\partial}{\partial y} \left( \frac{D_{*g}}{dt} u_o \right) = D_{*gt} \left( \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} \right) + \left( \frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} \right) \left( \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} \right)$ .

Thus,

$$D_{*gt} \left( \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + ry \right) = \frac{\partial w_1}{\partial z} + \text{curl} \mathbf{G} \quad (5.1.14)$$

As in the shallow water case, the evolution of the  $O(1)$  variables  $u_o, v_o$  are determined by the  $O(\varepsilon)$  variables. Since

$$w_1 = -\frac{1}{\Gamma_*} D_{*gt} \rho_o + \frac{S}{\Gamma_*} = -D_{*gt} \left( \frac{\rho_o}{\Gamma_*} \right) + \frac{S}{\Gamma_*} \quad (5.1.15)$$

$$\partial_z w_1 = -D_{gt} \left[ \frac{\partial}{\partial z} \left( \frac{\rho_o}{\Gamma_*} \right) \right] + \frac{\partial}{\partial z} \left( \frac{S}{\Gamma_*} \right)$$

Thus, we have the nondimensional QG P.V. equation:

$$D_{*gt} \left\{ ry + \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + \frac{\partial}{\partial z} \left( \frac{\rho_o}{\Gamma_*} \right) \right\} = \text{curl} \mathbf{G} + \partial_z \left( \frac{S}{\Gamma_*} \right) \quad (5.1.16)$$

or in the dimensional form

$$D_g q = S_q \quad (5.1.17a)$$

where

$$q = f_o + \beta y + \zeta + \partial_z \left( \frac{f_o \rho}{d\rho_s/dz} \right) \quad (5.1.17b)$$

$$S_q = \frac{1}{\rho_0} \mathbf{k} \cdot \nabla \times \mathbf{F} + \partial_z \left( \frac{f_o Q_o}{d\rho_s/dz} \right) \quad (5.1.17c)$$

$$\zeta = \partial_x v_g - \partial_y u_g = \nabla^2_H \psi \quad (5.1.17d)$$

$$D_g = \partial_t + u_g \partial_x + v_g \partial_y = \partial_t + J(\psi, \quad ) \quad (5.1.17e)$$

Since now

$$\rho = -\frac{\rho_o f_o}{g} \frac{\partial \psi}{\partial z}, \quad (5.1.18)$$

$$N^2 = -\frac{g}{\rho_0} \frac{d\rho_s}{dz}, \quad (5.1.19)$$

We have

$$\frac{\rho}{d\rho_s/dz} = -\frac{f_o}{N^2} \frac{\partial \psi}{\partial z}$$

The dimensional QGPV equation can be written in terms of  $\psi$  as:

$$\partial_t q + J(\psi, q) = S_q \quad (5.1.20a)$$

$$q = f_o + \beta y + \nabla^2_H \psi + \partial_z \left( \frac{f_o^2}{N^2} \frac{\partial \psi}{\partial z} \right) \quad (5.1.20b)$$

For unforced, adiabatic flow,  $q$  is conserved along the geostrophic flow (which is different from the original 3-D flow!)



#### 4. The Atmosphere Case

The QGPV equation in the atmosphere can be derived parallel to the oceanic equation. Typical scales are chosen as

$$u, v \sim U, \quad x, y \sim L, \quad w \sim U \frac{D}{L}, \quad z \sim D, \quad t \sim \frac{L}{U},$$

The Rossby number is  $\varepsilon = \frac{U}{f_o L}$ , the potential temperature is written as

$$\theta(x, y, z, t) = \Theta(z) + \theta'(x, y, z, t)$$

and the geostrophic potential high as

$$\phi(x, y, z, t) = \Phi(z) + \phi'(x, y, z, t)$$

where

$$\frac{d\Phi}{dz} = g \frac{\Theta(z)}{\theta_s}$$

$$\frac{d\phi'}{dz} = g \frac{\theta'}{\theta_s}$$

For large scale flows, the geopotential height anomaly is scaled as

$$\phi' \sim f_o L U$$

such that the pressure gradient is comparable to the Coriolis force at the first order.

The source/sink is also weak such that

$$\frac{1}{f_o U \rho} \mathbf{F} = \varepsilon \mathbf{G}, \quad \text{where } |\mathbf{G}| \leq O(1).$$

The local  $\beta$ -plane is adopted as

$$\frac{\beta L}{f_o} = \gamma \varepsilon, \quad O(r) \sim 1.$$

The two momentum equations are represented in dimensionless variables as:

$$\varepsilon \left\{ \frac{\partial u_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) u_* - \gamma y_* v_* \right\} - v_* = - \frac{\partial \phi_*}{\partial x_*} + \varepsilon G_x$$

$$\varepsilon \left\{ \frac{\partial v_*}{\partial x_*} + (\mathbf{u}_* \cdot \nabla_*) v_* + \gamma y_* u_* \right\} + u_* = - \frac{\partial \phi_*}{\partial y_*} + \varepsilon G_y$$

The mass equation is

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{1}{p} \frac{\partial}{\partial z_*} (p w_*) = 0$$

The scale of  $\theta'$  is estimated from the hydrostatic equation

$$\frac{\partial \phi}{\partial z} = g \frac{\theta}{\theta_s}$$

This gives

$$\theta' \sim \frac{\phi}{gD} \theta_s \sim \frac{f_o L U}{gD} \theta_s \sim \theta_s \left(\frac{L}{L_D}\right)^2 \varepsilon = \Gamma$$

where  $L_D^2 = \frac{gD}{f_o^2}$  is the external deformation radius. Thus,

$$\theta' \sim \Gamma \theta^* \quad \text{where } \theta^* \sim O(1),$$

and the hydrostatic balance is

$$\frac{\partial \phi_*}{\partial z_*} = \theta_*$$

With these scaling, the thermodynamic equation becomes

$$\begin{aligned} \frac{U\Gamma}{L} \left\{ \frac{\partial \theta_*}{\partial z_*} + (\mathbf{u}_* \cdot \nabla_*) \theta_* \right\} + \frac{UD}{L} w_* \frac{d\Theta}{dz} &= Q_a \\ \Rightarrow \varepsilon \left\{ \frac{\partial \theta_*}{\partial z_*} + (\mathbf{u}_* \cdot \nabla_*) \theta_* \right\} + \frac{UD}{f_o L \Gamma} w_* \frac{d\Theta}{dz} &= \frac{Q_a}{f_o \Gamma} \end{aligned}$$

The scale of  $\frac{d\Theta}{dz}$  can be anticipated to satisfy  $\frac{d\Theta}{dz} \geq \frac{1}{\varepsilon} \frac{\partial \theta}{\partial z} \gg \frac{\partial \theta}{\partial z}$ . For adiabatic flows, the solution is always along isentropics,  $\frac{d\theta}{dt} = 0$ . Since QG equations require  $\frac{w}{u} \leq \varepsilon \frac{D}{L}$ , the slope of the isentropic surface must be  $\leq \varepsilon \frac{D}{L}$

$$\frac{\partial \theta}{\partial x} / \frac{\partial \theta}{\partial z} \cong \alpha \leq \varepsilon \frac{D}{L}$$

Since  $\frac{\partial \Theta}{\partial x} = 0$ , we have

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta'}{\partial x} \sim \frac{\partial \theta'}{\partial z} \frac{D}{L}$$

and therefore

$$\begin{aligned} \frac{\partial \theta'}{\partial z} / \frac{\partial (\Theta + \theta')}{\partial z} &\leq \varepsilon \\ \frac{\partial \theta'}{\partial z} / \frac{\partial (\Theta)}{\partial z} &\leq \varepsilon \end{aligned}$$

Thus, we are led to the scale of

$$\frac{d\Theta}{dz} = \frac{\Gamma}{D\varepsilon} \Gamma_*(z) \quad \text{where } \Gamma_*(z) \leq O(1).$$

If we define a buoyancy frequency  $N(z)$ , then

$$N^2 = \frac{g}{\Theta_s} \frac{d\Theta}{dz}$$

The thermodynamic equation becomes

$$\varepsilon \left\{ \frac{\partial \theta_*}{\partial z_*} + (\mathbf{u}_* \cdot \nabla_*) \theta_* \right\} + w \Gamma_* = \varepsilon Q$$

where  $\varepsilon Q = \frac{Q_a}{f_o \Gamma}$  and  $Q \leq O(1)$  is consistent with leading order adiabatic.

The complete set of dimensionless equations is now (drop \*):

$$\begin{aligned} v - \frac{\partial \phi}{\partial x} &= \varepsilon \left\{ \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)u - ryv - G_x \right\} \\ -u - \frac{\partial \phi}{\partial y} &= \varepsilon \left\{ \frac{\partial v}{\partial t} + (\mathbf{u} \cdot \nabla)v + ryu - G_y \right\} \\ -w\Gamma_* &= \varepsilon \left\{ \frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta - Q \right\} \\ \frac{\partial \phi}{\partial z} - \theta &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{p} \frac{\partial}{\partial z}(pw) &= 0 \end{aligned}$$

The variables will be expanded as follows:

$$\begin{aligned} u &= u_o + \varepsilon u_1 + O(\varepsilon^2) \\ v &= v_o + \varepsilon v_1 + O(\varepsilon^2) \\ \phi &= \phi_o + \varepsilon \phi_1 + O(\varepsilon^2) \\ \theta &= \theta_o + \varepsilon \theta_1 + O(\varepsilon^2) \\ w &= w_o + \varepsilon w_1 + O(\varepsilon^2) \end{aligned}$$

The  $O(1)$  equations are

$$\begin{aligned} v_o &= \frac{\partial \phi_o}{\partial x}, \quad u_o = -\frac{\partial \phi_o}{\partial y}, \quad w_o = 0, \quad \frac{\partial \phi}{\partial z} = \theta_o \\ \frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} + \frac{1}{p} \frac{\partial}{\partial z}(pw_o) &= 0 \end{aligned}$$

In dimensional form, the geostrophic balance is:

$$\begin{aligned} v_g &= \frac{1}{f_o} \frac{\partial \phi'}{\partial x} \equiv \partial_x \psi \\ u_g &= -\frac{1}{f_o} \frac{\partial \phi'}{\partial y} \equiv -\partial_y \psi \end{aligned}$$

where  $\psi = \frac{\phi'}{f_o}$  is the geostrophy stream function. The hydrostatic equation is  $\frac{\partial \phi'}{\partial z} = \theta'$  or

$\frac{\theta'}{\theta_s} = \frac{f_o}{g} \frac{\partial \psi}{\partial z}$ . This leads to the thermal wind relationship

$$\frac{\partial v_g}{\partial z} = \frac{\partial^2 \psi}{\partial x \partial z} = \frac{g}{f_o \theta_s} \frac{\partial \theta'}{\partial x}, \quad \frac{\partial u_g}{\partial z} = -\frac{\partial^2 \psi}{\partial y \partial z} = -\frac{g}{f_o \theta_s} \frac{\partial \theta'}{\partial y}$$

At the next order, we have

$$v_1 - \frac{\partial \phi_1}{\partial x} = \frac{D_{*g}}{Dt} u_o - ryv_o - G_x$$

$$-u_1 - \frac{\partial \phi_1}{\partial y} = \frac{D_{*g}}{Dt} v_o + ry u_o - G_y$$

$$-w_1 \Gamma_* = \frac{D_{*g}}{Dt} \theta_o - Q$$

$$\frac{\partial \phi}{\partial z} - \theta = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{p} \frac{\partial (pw_1)}{\partial z} = 0$$

where  $\frac{D_{*g}}{Dt} = \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}$ . The vorticity equation can be derived as:

$$D_{*gt} \left( \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + ry \right) = \frac{1}{p} \frac{\partial}{\partial z} (pw_1) + \text{curl} \bar{G}$$

Since

$$w_1 = -\frac{1}{\Gamma_*} D_{*gt} \theta_o + \frac{Q}{\Gamma_*} = -D_{*gt} \left( \frac{\theta_o}{\Gamma_*} \right) + \frac{Q}{\Gamma_*}$$

$$\frac{1}{p} \frac{\partial}{\partial z} (pw_1) = -D_{gt} \left[ \frac{1}{p} \frac{\partial}{\partial z} (p \frac{\theta_o}{\Gamma_*}) \right] + \frac{1}{p} \frac{\partial}{\partial z} (p \frac{Q}{\Gamma_*})$$

We have the nondimensional QG P.V. equation:

$$D_{*gt} \left\{ ry + \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p \theta_o}{\Gamma_*} \right) \right\} = \text{curl} G + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p Q}{\Gamma_*} \right)$$

In the dimensional form

$$D_g q = S_q$$

where

$$q = f_o + \beta y + \zeta + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p f_o \theta'}{d\Theta/dz} \right), \quad S_q = \frac{1}{\rho} \mathbf{k} \cdot \nabla \times \mathbf{F} + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p f_o Q_a}{d\Theta/dz} \right)$$

$$\zeta = \partial_x v_g - \partial_y u_g = \nabla^2_H \psi, \quad D_g = \partial_t + u_g \partial_x + v_g \partial_y$$

Now, with

$$\theta' = \frac{\theta_s f_o}{g} \frac{\partial \psi}{\partial z}, \quad N^2 = \frac{g}{\theta_s} \frac{d\Theta}{dz}$$

and therefore  $\frac{\theta'}{d\Theta/dz} = \frac{f_o}{N^2} \frac{\partial \psi}{\partial z}$ , we have the QGPV equation for the atmosphere as:

$$\partial_t q + J(\psi, q) = S_q \tag{5.1.21a}$$

$$q = f_o + \beta y + \nabla^2_H \psi + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p f_o^2}{N^2} \frac{\partial \psi}{\partial z} \right). \tag{5.1.21b}$$

## Sec. 5.2: Rossby Waves in Stratified Flows

### 1. Dispersion Relationship

As in the shallow water case, we study small perturbations linearized on a mean zonal flow. The linearized QGPV equation is:

$$(\partial_t + U\partial_x)q' + v'Q_y = 0$$

where the mean and perturbation potential vorticity are

$$Q_y = \beta - \partial_{yy}U - \frac{1}{p} \partial_z \left[ \frac{pf_o^2}{N^2} \partial_z U \right],$$

$$q' = \partial_{xx}\psi + \partial_{yy}\psi + \frac{1}{p} \partial_z \left[ \frac{pf_o^2}{N^2} \frac{\partial\psi}{\partial z} \right]$$

We have used the atmospheric equation (5.1.21) and the oceanic equation can be recovered simply by setting  $p=p_0$ . The QGPV equation can be written in the perturbation streamfunction  $\psi$  as

$$(\partial_t + U\partial_x) \left\{ \partial_{xx}\psi + \partial_{yy}\psi + \frac{1}{p} \partial_z \left[ \frac{pf_o^2}{N^2} \partial_z \psi \right] \right\} + \partial_x \psi Q_y = 0 \quad (5.2.1)$$

We will assume the basic state is slowly varying, that is, the wave length in y, z directions are short relative to the scales at which  $U$ ,  $N^2$  and  $Q_y$  vary. (But, we don't need to assume  $p(z)$  slowly varying!). Assuming the solution of the form

$$\psi(x, y, z, t) = \text{Re} \left[ \Psi(y, z) e^{i\theta + \frac{z}{2H}} \right]$$

where

$$\theta = k(x - ct) + \int^y l(y') dy' + \int^z m(z') dz'$$

We have approximately

$$\partial_{xx}\psi \approx -k^2\psi \quad , \quad \partial_{yy}\psi \approx -l^2\psi$$

and

$$\begin{aligned} \frac{1}{p} \partial_z \left( \frac{pf_o^2}{N^2} \partial_z \psi \right) &\cong \frac{f_o^2}{N^2} \frac{1}{p} \partial_z (p \partial_z \psi) \\ &\approx \frac{f_o^2}{N^2} e^{\frac{z}{H}} \partial_z \left( e^{-\frac{z}{H}} \partial_z \psi \right) \\ &\approx -\frac{f_o^2}{N^2} \left( m^2 + \frac{1}{4H^2} \right) \Psi e^{i\theta} e^{\frac{z}{H}} \\ &= -\frac{f_o^2}{N^2} \left( m^2 + \frac{1}{4H^2} \right) \psi \end{aligned}$$

Therefore, (5.2.1) gives the dispersion relationship as

$$(U - c)[k^2 + l^2 + \frac{f_o^2}{N^2}(m^2 + \frac{1}{4H^2})] - Q_y = 0.$$

That is

$$c = U - \frac{Q_y}{k^2 + l^2 + \frac{f_o^2}{N^2}(m^2 + \frac{1}{4H^2})} \quad (5.2.2)$$

for the atmosphere. For the ocean, we can set  $H$  infinitely large (incompressible), so that

$$c = U - \frac{Q_y}{k^2 + l^2 + \frac{f_o^2}{N^2}m^2} \quad (5.2.3)$$

The relation between the shallow water Rossby waves and the baroclinic Rossby waves here are readily seen if we make

$$L_{Dm}^2 = \frac{N^2}{f^2(m^2 + \frac{1}{4H^2})} \quad (5.2.4)$$

The dispersion relationship can be put exactly the same form as the shallow water Rossby wave

$$c = U - \frac{Q_y}{k^2 + l^2 + L_{Dm}^{-2}}$$

The  $L_{Dm}$  is the deformation radius for baroclinic flows with a vertical wave number  $m$ . Since  $L_{Dm} \sim 1/m$ , the deformation radius increases and the wave speed faster for smaller  $m$  (or larger vertical scale), and vice versa. For typical atmospheric stratification, we have  $L_{D1} \sim 1000 \text{ km}$ , while for typical oceanic stratification, we have  $L_{D1} \sim 50 \text{ km}$ .

## **2. Group Velocity and Vertical Propagation**

For a given  $k, l$ , the dispersion relationship gives the vertical wave number

$$m^2 = \frac{N^2}{f_o^2} \left[ \frac{Q_y}{U - c} - k^2 - l^2 \right] - \frac{1}{4H^2} \quad (5.2.5)$$

When the RHS  $> 0$ ,  $m$  is real, and the Rossby wave propagate vertically; When the RHS  $< 0$ ,  $m$  becomes imaginary, and the waves are trapped vertically.

For propagation waves (real  $m$ ), the group velocity can be calculated as

$$\begin{aligned} C_{gx} &= \frac{\partial \omega}{\partial k} = U + \Delta \left[ k^2 - l^2 - \frac{f_o^2}{N^2} \left( m^2 + \frac{1}{4H^2} \right) \right] \\ C_{gy} &= \frac{\partial \omega}{\partial l} = 2kl\Delta \\ C_{gz} &= \frac{\partial \omega}{\partial m} = 2 \frac{f_o^2}{N^2} km\Delta \end{aligned} \quad (5.2.6)$$

where

$$\Delta = Q_y [k^2 + l^2 + \frac{f_o^2}{N^2} (m^2 + \frac{1}{4H^2})]^{-2} = \frac{(U - c)^2}{Q_y}$$

Note 1: If  $l$  is replaced by  $\frac{f_o^2}{N^2} m$ , we have  $C_{gy}$  the same as  $C_{gz}$ . Therefore, mathematically, the  $y$  direction and  $z$  direction are very similar for Rossby waves. However, later, we will see that the physical meaning of the group velocity in these two directions differ dramatically.

In the case of  $U=0$ , we have  $Q_y = \beta > 0$ , we have  $C_{px} = \frac{\omega}{k} < 0$ , so the wave always propagates westward. Define the phase velocity as

$$\mathbf{C}_p = (C_{px}, C_{py}, C_{pz}) \equiv \left( \frac{\omega}{k}, \frac{\omega}{l}, \frac{\omega}{m} \right)$$

We see from (5.2.2) and (5.2.6) that

$$\text{sign}(C_{py}) = -\text{sign}(l), \text{ but } \text{sign}(C_{gy}) = +\text{sign}(l)$$

$$\text{sign}(C_{pz}) = -\text{sign}(m), \text{ but } \text{sign}(C_{gz}) = +\text{sign}(m)$$

Thus, the phase and group velocity are in the opposite directions.

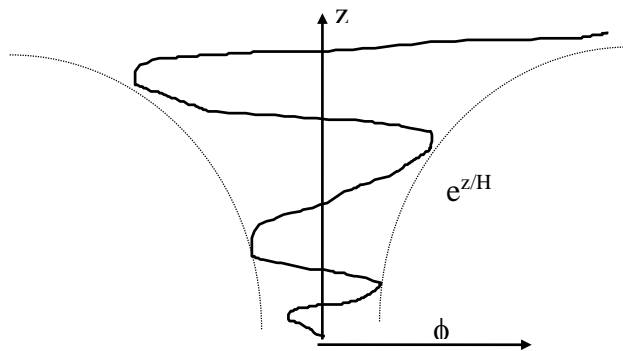
It should be pointed out that in the atmosphere, even if  $m$  is real (when the basic state allows),  $\psi$  does not vary with height simply as  $e^{imz}$ ; instead, its amplitude increases with height as

$$\psi \sim e^{imz} e^{\frac{z}{2H}}$$

or the energy increases with height as

$$|\psi|^2 \sim e^{z/H} \sim 1/p$$

The amplification of the streamfunction with height is caused by the reduction of atmospheric pressure.



Finally, for stationary forcing (topography or large scale heating/cooling)  $c=0$ . Eqn.(5.2.5) shows that only those largest scale waves (smaller  $k, l$ ) can propagate vertically in the westerly wind ( $U > 0$ ) (real  $m$ ). This has been used to explain the observed stratosphere. Stratosphere disturbances are believed to originate from the troposphere. Observations show that the mid- and high latitude stratosphere is dominated by disturbances at planetary scales (wave number 1, 2, 3), although the most energetic disturbances in the troposphere are at higher wave numbers (6 and 7). This is because only those very long waves can propagate into the stratosphere according to (5.2.2).

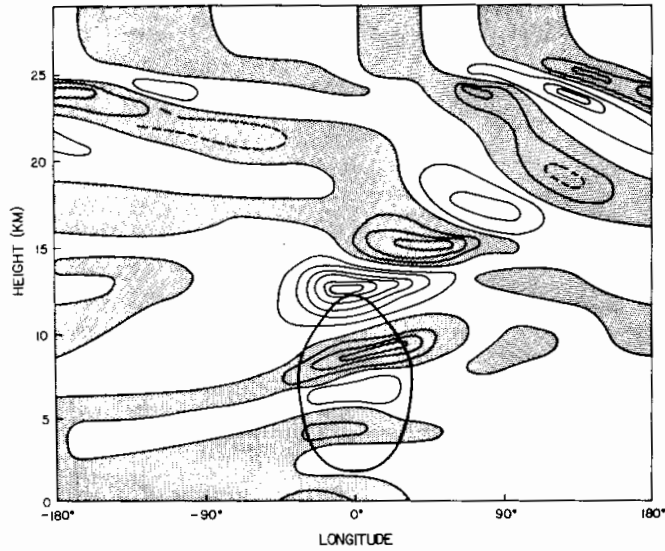


Fig. 11.13. Longitude–height section of the meridional velocity perturbation at the equator as found by Holton (1972, Fig. 9) for an antisymmetric source of diabatic heating that oscillates with an amplitude exceeding  $4 \text{ K day}^{-1}$  inside the heavy line. Contours are at  $2 \text{ m s}^{-1}$  intervals. The waves produced are mainly mixed planetary–gravity waves. The mean wind varies with height with a maximum eastward velocity of  $8 \text{ m s}^{-1}$  at 21 km, zero velocity at 25 km, and westward velocity above that level.

Fig.5.1 Vertical propagation of atmospheric waves

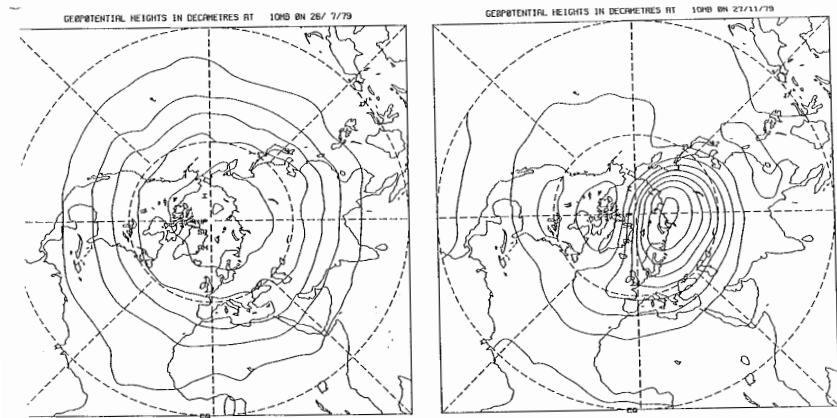
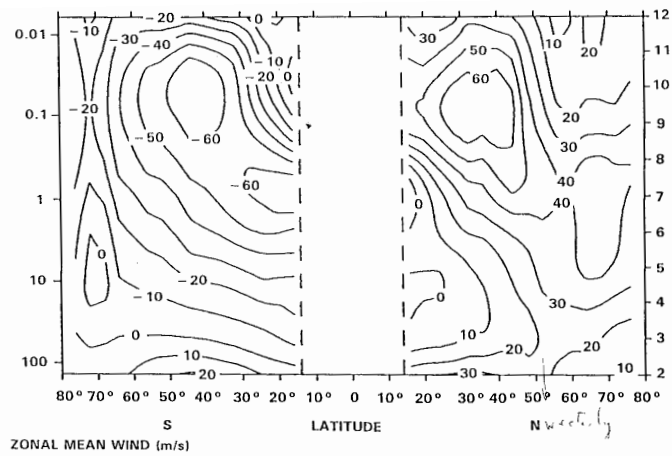


Fig.5.2 Geopotential height anomaly in the stratosphere



## Sec. 5.3: Vertical Normal Modes

### 1. Vertical Modes in the Ocean

Consider an ocean with  $U=0$  and  $N$  uniform. The mean PV gradient is therefore  $Q_y = \beta$ . The linearized QGPV equation is:

$$\partial_t [\partial_{xx} + \partial_{yy} + \frac{f_o^2}{N^2} \partial_{zz}] \psi + \beta \partial_x \psi = 0 \quad (5.3.1)$$

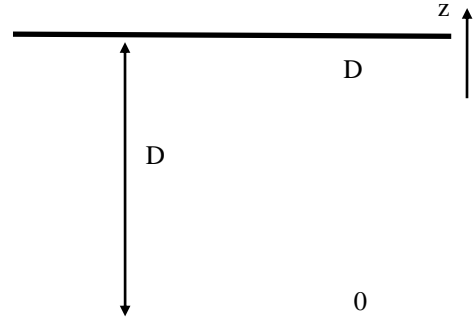
On the bottom (assume flat)

$$w(x,y,0,t) = 0 \quad (5.3.2)$$

At the top  $z=D+\eta(x,y,t)$

We have a rigid lid

$$w(x,y,D,t) = 0. \quad (5.3.3)$$



To write the vertical boundary condition in terms of  $\psi$ , we resort to the thermodynamic equation:

$$\partial_t \rho' + w' \frac{d\rho_s}{dz} = 0$$

The hydrostatic balance gives:

$$\frac{\partial \psi}{\partial z} = - \frac{g}{f_o} \frac{\rho'}{\rho_0} \quad (5.3.4)$$

We then have

$$w = - \frac{\partial_t \rho'}{\frac{d\rho_s}{dz}} = \frac{f_o \rho_0}{g} \frac{\partial^2 \psi}{\partial t \partial z} = - \frac{f_o}{N^2} \partial_{tz} \psi \quad (5.3.5)$$

Thus, the vertical boundary conditions (5.3.2) and (5.3.3) become

$$\partial_{tz} \psi = 0 \quad \text{at } z=0, D \quad (5.3.6)$$

returning to the QGPV equation. We look for separable solutions of the form

$$\psi = \phi(z) \Psi(x, y) e^{-i\omega t}$$

Substitute this into the QGPV equation, we have the equation for the vertical structure as

$$\frac{d}{dz} \left[ \frac{f_o^2}{N(z)^2} \frac{d\phi}{dz} \right] = -\lambda^2 \phi \quad (5.3.7)$$

and the equation for the horizontal structure as

$$-i\omega [\partial_{xx} \Psi + \partial_{yy} \Psi - \lambda^2 \Psi] + \beta \frac{\partial \Psi}{\partial x} = 0 \quad (5.3.8)$$

The vertical boundary conditions (5.3.6) becomes

$$\frac{d\phi}{dz} = 0 \quad \text{on } z=0, D \quad (5.3.9)$$

Therefore, (5.3.7) and (5.3.9) form an eigenvalue problem. The eigenvalues are real. Indeed, notice (5.3.9),  $\int_0^D \phi \frac{N^2}{f_o^2} \times (5.3.7)$  gives

$$\begin{aligned} -\lambda^2 \int_0^D \frac{N^2}{f_o^2} \phi^2 dz &= \int_0^D \phi \frac{d^2 \phi}{dz^2} dz = \int_0^D \left[ \frac{d}{dz} \left( \phi \frac{d\phi}{dz} \right) - \left( \frac{d\phi}{dz} \right)^2 \right] dz \\ &= \phi \frac{d\phi}{dz} \Big|_0^D - \int_0^D \left( \frac{d\phi}{dz} \right)^2 dz = - \int_0^D \left( \frac{d\phi}{dz} \right)^2 dz. \end{aligned}$$

Therefore, the eigenvalue is

$$\lambda^2 = \frac{\int_0^D \left( \frac{d\phi}{dz} \right)^2 dz}{\int_0^D \frac{N^2}{f_o^2} \phi^2 dz} > 0$$

In the case of a uniform  $N$ , the eigenfunctions and eigenvalues can be easily solved as

$$\phi_m(z) = \cos\left(\frac{N}{f_o} \lambda_m z\right) \quad (5.3.10)$$

$$\frac{N}{f_o} \lambda_m = \frac{m\pi}{D}, \quad m = 0, 1, 2, \dots \quad (5.3.11)$$

Substitute them into (5.3.8), we have the dispersion relationship

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{f_o^2}{N^2} \left(\frac{m\pi}{D}\right)^2}$$

Thus, for each vertical mode,  $m$ , the dispersion relationship is exactly the same as that for the shallow water Rossby wave, provided that we replace the effective deformation radius for each mode as:

$$L_{Dm}^2 = \frac{(ND)^2}{(f_o m \pi)^2}. \quad (5.3.12)$$

The deformation radius vanishes, with an increasing  $m$ .

The correspondence of the deformation radius in (5.3.12) with that in the 1.5-layer model (1.5.4) can be readily seen below. Since

$$N^2 = -\frac{g}{\rho_o} \frac{d\rho_s}{dz} \sim g \frac{\Delta\rho}{\rho_o} \frac{1}{D} = g' \frac{1}{D},$$

We have

$$L_{Dm}^2 = \frac{g'}{D} \frac{D^2}{f_o^2 (m\pi)^2} = \frac{g' D}{f_o^2} \frac{1}{(m\pi)^2} = \frac{g' D_m}{f_o^2}$$

where

$$D_m = \frac{D}{(m\pi)^2}$$

is the equivalent depth. Thus, each baroclinic mode propagate exactly as a 1.5 layer model Rossby wave with an equivalent depth of  $D_m$ . (Some people also use the expression of  $L_{Dm}^2 = \frac{g\hat{D}_m}{f_o^2}$ , such that the equivalent depth is  $\hat{D}_m = \frac{\Delta\rho}{\rho} D_m$ )

The  $L_{Dm}$  is called the internal (baroclinic) deformation radius. This gives a close analogy between shallow water dynamics and the stratified dynamics. In the ocean,  $\frac{L_{D1}}{L_{Do}} = \frac{\Delta\rho}{\rho} \ll 1$ .

The vertical structure of the normal modes is further discussed below. The  $m=0$  mode is the barotropic mode or external mode (deformation radius infinitely large in the absence of free surface elevation here). The velocity does not have shear in the vertical direction and there is no density perturbation for this mode.

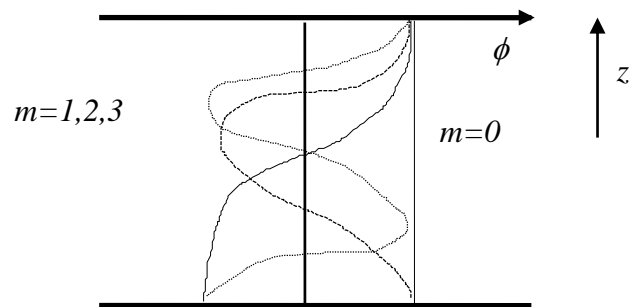
The  $m \geq 1$  mode is the  $m^{th}$  baroclinic mode or internal mode. These modes have  $m$  node points in the velocity field and are all accompanied with density perturbations. For all the baroclinic modes, the vertically integrated net transport vanishes.

This can be shown directly from (5.3.7)

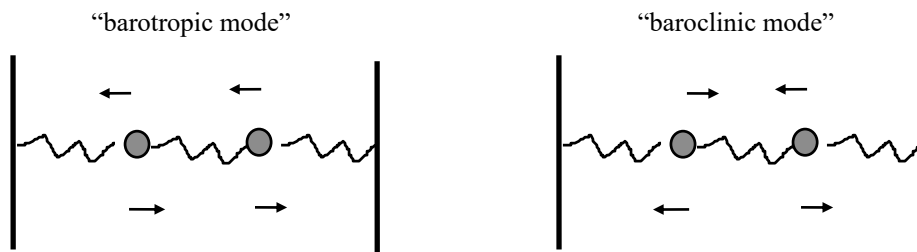
and (5.3.9). Integrate (5.3.7) from

$$z=0 \text{ to } D \text{ gives } \lambda_m^2 \int_0^D \phi_m(z') dz' = 0$$

$$\text{and therefore } \int_0^D \phi_m(z') dz' = 0, \text{ if } \lambda_m \neq 0.$$



**Note 1:** Equivalent particle examples of external and internal modes. Consider two balls connected by a spring. There are two possible normal modes. The first has both balls moving in the same direction, as if there is only one ball. This is the "barotropic mode". The second has the two balls always moving in the opposite directions. This is the "baroclinic mode".

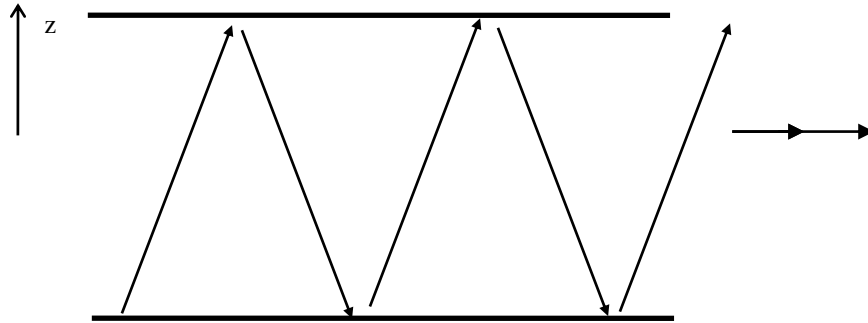


When more balls are added, there are more freedoms and more "modes".

In reality,  $N(z)$  is not uniform at all (Fig.5.3). Analytical solution becomes usually impossible. Nevertheless, for slowly varying  $N(z)$ , we can still use the WKB method such that, with  $U=0$ , (5.2.5) becomes:

$$m^2 = -\frac{N^2(z)}{f^2} \left( \frac{\beta}{C} + k^2 + l^2 \right)$$

for the oceanic case. If  $m$  is real at any height, it is real at all height, although  $N(z)$  may change. So there is no internal reflection. The normal mode is caused by reflection at the top and bottom boundaries.



After a couple of reflections, normal mode is established in the  $z$  direction. The establishment of the normal mode is similar to the normal mode in the case of horizontal boundaries. The key is that the wave energy is trapped within a finite region.

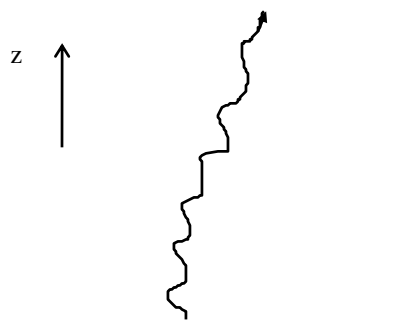
The most dramatic change of  $N$  is in the oceanic thermocline. The WKB solution shows that  $m(z)$  changes, small for small  $N$ , but large for large  $N$ . (Fig.5.4).

Indeed, for a general  $N(z)$ , the vertical eigenvalue problem (5.3.7) and (5.3.9) can be solved numerically to give  $\lambda_m$ . Then, the horizontal structure satisfies (5.3.8), which is the same as the shallow water Rossby wave, except for replacing the deformation radius by  $L_D^2 = \lambda_m^{-2}$ . The Rossby wave of the  $m^{\text{th}}$  mode has the dispersion relationship

$$\omega = \frac{-\beta k}{k^2 + l^2 + \lambda_m^2}$$

## 2. Atmospheric Case

The normal mode in the atmosphere is much more complicated, because of the lack of a upper boundary. It turns out that the normal mode usually doesn't exist in the atmosphere. This is not hard to imagine, because, in the absence of a reflective top boundary, a normal mode can't be established.



Mathematically, one can see this crudely here. If the vertical mode equation allows the solution

$$\phi \sim Ae^{imz} + Be^{-imz}$$

The absence of an energy source from above requires that the wave energy radiates upward only. This selects  $B=0$ . Furthermore, at the bottom,  $\partial_z \phi = 0$  determines that  $A = 0$ . Therefore, there is no normal mode. However, normal modes may exist when a strong shear of  $U(z)$  produces internal reflections.

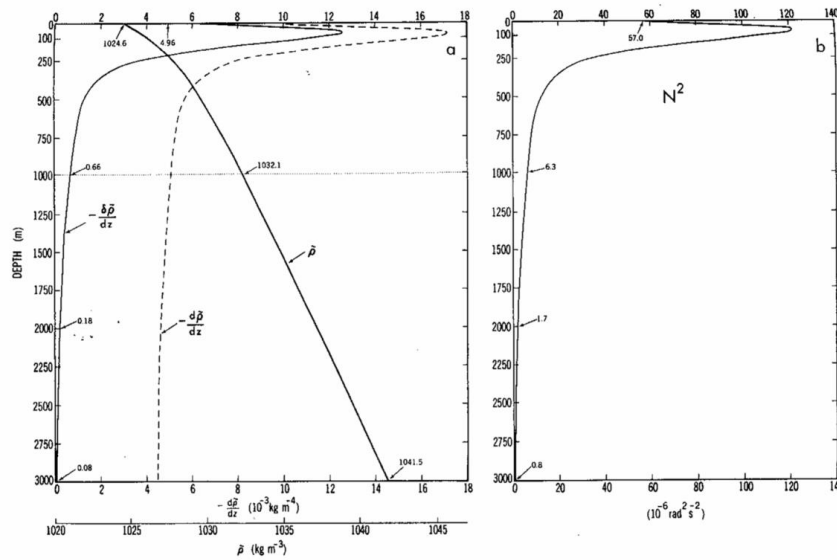


FIGURE 8.16. Vertical profiles of the global-mean density in  $\text{kg m}^{-3}$  and two measures of the mean static stability (see the text) in  $10^{-3} \text{ kg m}^{-4}$  (a), and a vertical profile of the global mean Brunt-Väisälä frequency squared (b) in  $10^{-6} \text{ rad}^2 \text{ s}^{-2}$ .

Fig.5.3 Vertical profiles of  $N$  in the atmosphere and ocean

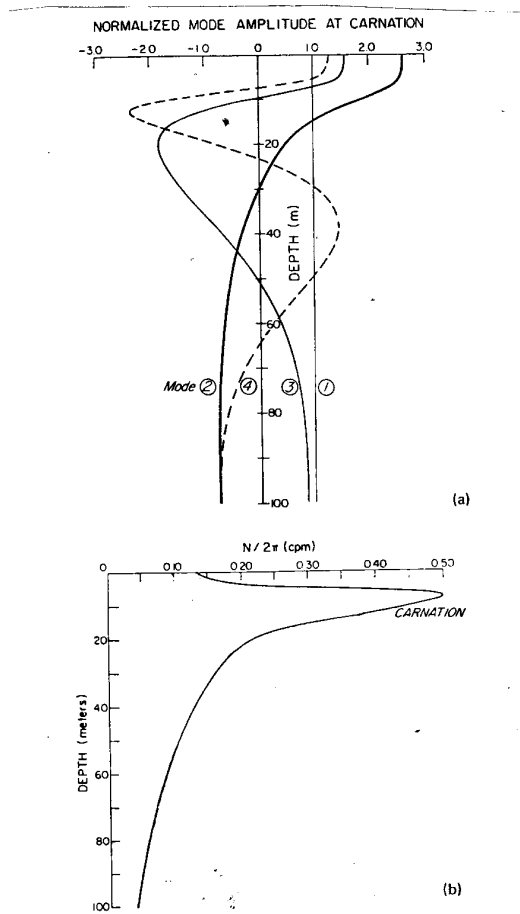


Figure 6.12.1 (a) The barotropic and first three baroclinic modes, as calculated by Kundu, Allen, and Smith (1975) for (b) the distribution of  $N$  observed at ocean station Carnation, off the Oregon coast.

Fig.5.4 Normal modes in the presence of realistic oceanic stratification.

## Sec 5.4: The Eliassen-Palm Theorem

### 1. E-P theorem:

The shallow water E-P theorem in section 2.6 can be generalized to the stratified fluid. The QGPV equation of the atmosphere is

$$D_g q = (\partial_t + u\partial_x + v\partial_y)q = S_q$$

Consider a basic state

$$U = U(y, z), \quad \Theta = \Theta(z), \quad \Psi = \Psi(y, z), \quad \phi = f_o \Psi(y, z) + \Phi(z)$$

where

$$\frac{\partial \Psi}{\partial y} = -U(y, z), \quad \frac{d\Phi}{dz} = \frac{g\bar{T}(z)}{T_s} = \frac{g}{T_s} \left(\frac{p}{p_*}\right)^k \Theta$$

and the mean temperature field

$$T = T(y, z) + \bar{T}(z) = T(y, z) + \left(\frac{p}{p_*}\right)^k \Theta(z)$$

where the basic state satisfies the thermal wind relationship

$$-\frac{g}{T_s} \frac{\partial T}{\partial y} = f_o \frac{\partial U}{\partial z}$$

The mean QGPV is therefore:

$$Q(y, z) = f_o + \beta y + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p f_o^2}{N^2} \frac{\partial \Psi}{\partial z} \right)$$

and the mean PV gradient is

$$Q_y(y, z) = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{1}{p} \frac{\partial}{\partial z} \left( \frac{p f_o^2}{N^2} \frac{\partial U}{\partial z} \right)$$

Write

$$\psi = \Psi(y, z) + \psi'(x, y, z, t)$$

where  $\psi' \ll \Psi$ , is a small perturbation. The QGPV equation can be linearized as

$$(\partial_t + U\partial_x)q' + v'Q_y = S'_q \quad (5.4.1)$$

Multiplying the equation by  $pq'/Q_y$ , we have

$$(\partial_t + U\partial_x) \frac{pq'^2}{2Q_y} + pv'q' = pq'S'_q / Q_y \quad (5.4.2)$$

Since

$$v' = \partial_x \psi', \quad q' = \partial_{xx} \psi' + \partial_{yy} \psi' + \frac{1}{p} \partial_z \left( p \frac{f_o^2}{N^2} \partial_z \psi' \right)$$

$$v' \partial_{xx} \psi' = \frac{1}{2} \partial_x [(\partial_x \psi')^2]$$

$$\begin{aligned}
v' \partial_{yy} \psi' &= \partial_y [\partial_x \psi' \partial_y \psi'] - \frac{1}{2} \partial_x [(\partial_y \psi')^2] \\
v' \frac{\partial}{\partial z} [p \frac{f_o^2}{N^2} \partial_z \psi'] &= \partial_z [p \frac{f_o^2}{N^2} \partial_x \psi' \partial_z \psi'] - \partial_{xz}^2 \psi' \frac{p f_o^2}{N^2} \partial_z \psi' \\
&= \partial_z [p \frac{f_o^2}{N^2} \partial_x \psi' \partial_z \psi'] - \frac{1}{2} \partial_x [\frac{p f_o^2}{N^2} (\partial_z \psi')^2]
\end{aligned}$$

$pv'q'$  can be written in the form of flux divergence

$$pv'q' = \nabla \cdot \mathbf{F} \quad (5.4.3)$$

Here, the flux  $\mathbf{F}$  is the generalized E-P flux

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \frac{p}{2} [(\psi'_x)^2 - (\psi'_y)^2 - \frac{f_o^2}{N^2} (\partial_z \psi')^2] \\ p \partial_x \psi' \partial_y \psi' \\ \frac{p f_o^2}{N^2} \partial_x \psi' \partial_z \psi' \end{bmatrix} = \begin{bmatrix} \frac{p}{2} [u'^2 - v'^2 - (N \frac{\theta'}{d\Theta/dz})^2] \\ -pu'v' \\ \frac{p f_o v' \theta'}{d\Theta/dz} \end{bmatrix} \quad (5.4.4)$$

and

$$N^2 = \frac{g}{\Theta_s} \frac{d\Theta}{dz}, \quad \partial_z \psi' = \frac{\theta'}{f_o} \frac{N^2}{d\Theta/dz}$$

The perturbation PV equation (5.4.2) can be written in the wave activity equation:

$$(\partial_t + U \partial_x) A + \nabla \cdot \mathbf{F} = S \quad (5.4.5)$$

where

$$A = \frac{\frac{p}{2} q'^2}{Q_y} \quad (5.4.6)$$

is the wave activity, and  $S = S_q pq' / Q_y$ . The conventional E-P equation is the zonal mean of the generalized E-P equation.

$$\partial_t \bar{A} + \nabla \cdot \bar{\mathbf{F}} = \bar{C} \quad (5.4.7)$$

where

$$\bar{A} = \frac{p \frac{1}{2} \overline{q'^2}}{Q_y}, \quad \bar{\mathbf{F}} = \begin{bmatrix} \bar{F}_y \\ \bar{F}_z \end{bmatrix} = \begin{bmatrix} -p \overline{u'v'} \\ \overline{v' \theta'} \\ p f_o \frac{d\Theta/dz}{} \end{bmatrix} \quad (5.4.8)$$

This gives the Eliasson - Palm Theorem.

For (i) steady amplitude  $\partial_t A = 0$ , and (ii) conservative  $S = 0$ , the E-P flux  $\mathbf{F}$  is non-divergent. (Therefore,  $\mathbf{F}$  can't originate from nowhere and end in nowhere, like the mass flux of an incompressible fluid)



Note 1: For the ocean:

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} (\psi'_x)^2 - (\psi'_y)^2 - \frac{f_o^2}{N^2} (\partial_z \psi')^2 \\ \partial_x \psi' \partial_y \psi' \\ \frac{f_o^2}{N^2} \partial_x \psi' \partial_z \psi' \end{bmatrix} = \begin{bmatrix} u'^2 - v'^2 - (N \frac{\rho'}{d\rho_s/dz})^2 \\ -u'v' \\ \frac{f_o v' \rho'}{d\rho_s/dz} \end{bmatrix}$$

## 2. Wave Activity Flux and Group Velocity

For almost-plane waves, under the WKB assumption, the solution can be assumed of the form

$$\psi'(x, y, z, t) = \text{Re} \left[ \Psi(y, z) e^{i\theta + \frac{z}{2H}} \right]$$

where  $\theta = k(x - ct) + \int^y l(y') dy' + \int^z m(z') dz'$ . We can derive the wave activity as

$$A = \frac{p}{2} \overline{q'^2} / Q_y = \frac{1}{4\Delta} |\Psi|^2$$

where we have used

$$q' \approx \left[ k^2 + l^2 + \frac{f_o^2}{N^2} \left( m^2 + \frac{1}{4H^2} \right) \right] \psi'.$$

Similarly, the flux is

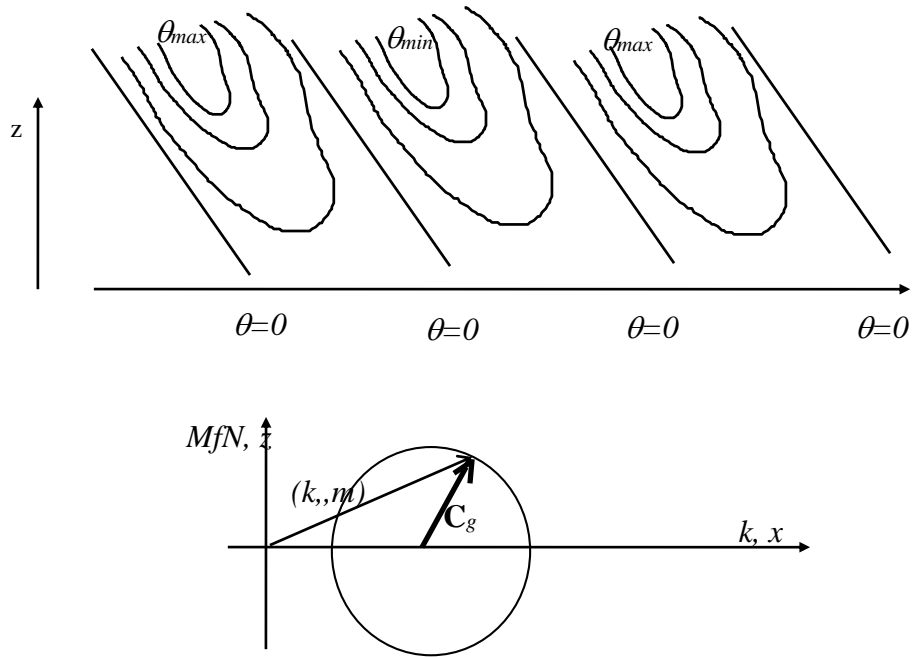
$$\begin{aligned} \overline{F}_y &= p \overline{\partial_x \psi' \partial_y \psi'} \approx \frac{1}{2} kl |\Psi|^2 \\ \overline{F}_z &= p \frac{f_o^2}{N^2} \overline{\partial_x \psi' \partial_z \psi'} \approx \frac{1}{2} \frac{f_o^2}{N^2} \text{Re} \left[ ik \Psi \left[ \left( im + \frac{1}{2H} \right) \Psi \right]^* \right] \approx \frac{1}{2} \frac{f_o^2}{N^2} km |\Psi|^2 \\ \overline{\mathbf{F}} &= (\overline{F}_y, \overline{F}_z) = \left[ 2kl\Delta, 2 \frac{f_o^2}{N^2} km\Delta \right] A = (C_{gy}, C_{gz}) A \end{aligned} \quad (5.4.9)$$

## 3 Vertical Propagation and Meridional Heat Transport

The vertical component of the E-P flux is directly related to the meridional heat flux

$$\overline{F}_z = p \frac{f_o^2}{N^2} \overline{\partial_x \psi' \partial_z \psi'} = \frac{\theta'}{f_o} \frac{N^2}{d\Theta/dz} \overline{v' \theta'} \propto \overline{v' \theta'} \propto km |\Psi|^2 \quad (5.4.10)$$

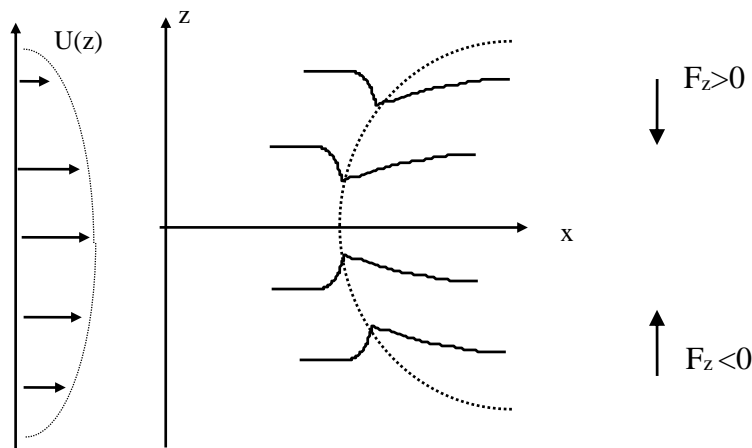
An upward E-P flux ( $\overline{F}_z > 0$ ) corresponds to a northward heat transport, and vice versa. In the atmosphere, Rossby waves are usually forced from the surface and propagate upward ( $\overline{F}_z > 0$ ). This corresponds to a westward tilt ( $km > 0$ ) and should transport heat poleward. In addition, waves are also caused by baroclinic instability (Chapter 6). The unstable waves also tilt westward and transport heat poleward.



The northward heat transport by Rossby waves contributes to the major part of the atmospheric poleward heat transport in the midlatitude region. Interestingly, in the ocean, the wind forces Rossby waves at the surface and therefore downward. These waves will transport heat equatorward, against the mean gradient.

**4. Applications**

Case 1: Wave-mean flow interaction in vertically sheared flow. Assuming a westerly wind with a maximum speed in the middle level. In the lower half, the westward tilting trough produces an upward E-P flux. The accompanied northward heat transport is down the mean temperature gradient ( $T_y < 0$  for  $U_z > 0$ ) and therefore tends to reduce the mean temperature gradient. The perturbation grows by extracting APE from the mean APE. Similar discussions show that the perturbation in the upper half is also unstable. Alternatively, the E-P flux converges, increasing the wave activity at the expense of the mean flow strength. ( $\partial_t A$  increases and  $\partial_t U$  decreases).



Case 2: Vertical propagation of the atmospheric Rossby waves (see the end of last section).

The amplitude increases for a vertically propagation wave with height inversely proportional to pressure. For a wave packet originate at the surface (1000mb) propagating into the stratosphere (10mb), its amplitude increases by 10 times. This can be seen using the E-P theorem. For steady, conservation waves,

$$\nabla \cdot \mathbf{F} = 0.$$

For plane waves,  $F_y$  is independent of  $y$ , so that

$$\partial_z \overline{F_z} = -\partial_y \overline{F_y} = 0,$$

or

$$\partial_z \left[ \frac{p f_o^2}{N^2} \overline{\psi'_x \psi'_z} \right] = 0 \quad (5.4.11)$$

If  $N$  is constant,

$$\partial_z \left[ p \overline{\psi'_y \psi'_z} \right] = 0 \quad (5.4.12)$$

Now, if

$$\psi' = \text{Re}[\Gamma(y, z)e^{i\theta}]$$

where  $\Gamma = \Psi e^{\frac{z}{2H}}$ . We have  $\partial_x \psi' = \text{Re}(ik\Gamma e^{i\theta})$ ,  $\partial_z \psi' = \text{Re}[(im + \frac{1}{2H})\Gamma e^{i\theta}]$ . Therefore,

$\overline{\partial_x \psi' \partial_z \psi'} = \frac{1}{2} km |\Gamma|^2$ . Thus, (5.4.12) gives

$$\partial_z (p |\Gamma|^2) = 0$$

In reality, the amplitude can be changed by dissipation, nonlinearity, wave refraction, etc.

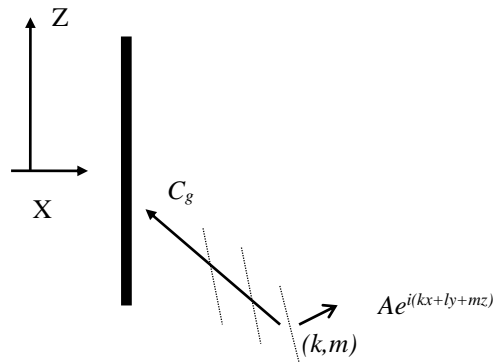
**Questions for Chapter 5**

**Exercises for Chapter 5**

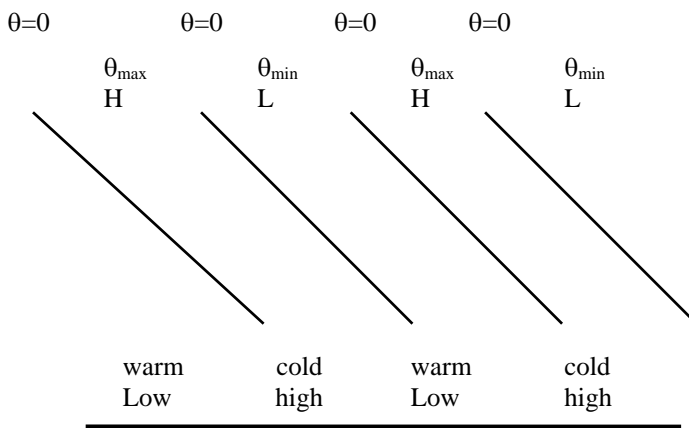
**E5.1.** (Vertical Rossby wave propagation diagram) The basic state is motionless and has a constant Brunt-Vasara frequency  $N$ . Baroclinic Rossby waves have the form of  $e^{i(kx+ly+mz)}$

a) Discuss the mathematical similarity between the vertical propagation of stratified baroclinic Rossby waves and the meridional propagation of shallow water Rossby waves. Plot the wave vector and direction of the group velocity in the wave number  $(k,m)$  plane (or the dispersion diagram circle).

b) In light of (a), consider an upward/westward propagating baroclinic Rossby wave that is incident on a vertical wall (or tall mountain). What will be the direction of the reflected Rossby wave? What will be the wave phase pattern?



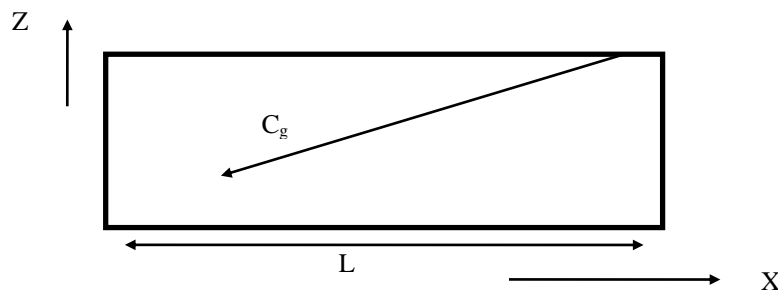
(c) If a baroclinic system tilts westward with height, what is the direction of Rossby wave energy propagation?



What kind of weather synoptic system does this correspond to? Which direction does this system transport heat flux? What is the direction of the E-P flux in the vertical direction? (For simplicity, you can assume an infinite scale height, or incompressibility).

**E5.2.** (Wind forced stratified ocean) A stratified linear ocean is forced by a spatially uniform Ekman pumping on the surface with a frequency  $\sigma$ . The ocean basin has a zonal scale  $L$  (such that the wave number is  $k=2\pi/L$ ) that is much longer than the internal deformation radius:

- (a) Find the direction of the downward group velocity.
- (b) What is the direction of the group velocity when the forcing frequency approaches zero (the limit of steady forcing)?
- (c) In light of (b), is it possible to have subsurface motion under a steady wind forcing?
- (d) Is Sverdrup relation valid in the limit of a steady wind?
- (e) What is the implication of (c) and (d)?



**E5.3** (Non-Doppler shift effect) In a stratified ocean, we will consider planetary scale perturbations that are governed by the potential vorticity equation

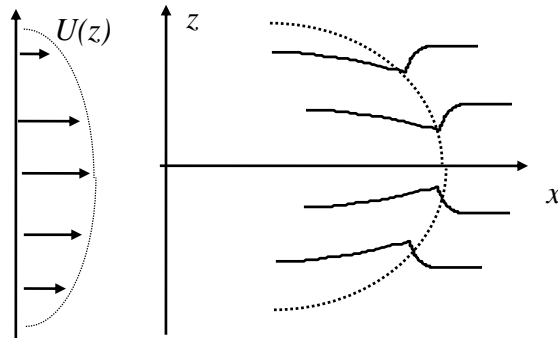
$$\partial_t \left[ \partial_z \left( \frac{f_o^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] + \beta \partial_x \psi + J \left[ \psi, \partial_z \left( \frac{f_o^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] = 0.$$

We project the streamfunction on vertical modes:  $\psi = \sum_{m=0}^{\infty} \Psi_m(x, y, t) \phi_m(z)$  where the  $m^{th}$  vertical mode  $\phi_m$  is determined by the eigenvalue equation (5.3.7) as

$$\frac{d}{dz} \left[ \frac{f_o^2}{N(z)^2} \frac{d\phi_m(z)}{dz} \right] = -\lambda_m^2 \phi_m(z).$$

- (a) If the flow is projected only on a single vertical mode  $m=M$ , what is the advection term in the potential vorticity equation.
- (b) In light of (a), how do you interpret the perfect non-Doppler-shift effect of the planetary Rossby wave in the shallow water or the 1.5-layer model?
- (c) If the flow is projected on more than one vertical modes, show that the  $m^{th}$  mode of the flow only advects the part of the stretching vorticity that excludes mode  $m$ .
- (d) Based on (c), under what condition, Rossby waves will be advected by mean flow (or Doppler-shift occurs) in a general continuously stratified ocean?

**E5.4:** (Wave-mean flow interaction of baroclinic waves). Based on the wave activity equation and E-P flux (5.4.7) and (5.4.8), discuss the wave-mean flow interaction of the following disturbances in a westerly shear flow.



- (a) Will the disturbance grow or decay? Will the mean flow intensify or weaken?
- (b) Discuss the difference and similarity from the corresponding barotropic case (in section 2.6) of negative viscosity.