Chapter 6: Instability

Sec. 6.1: Instability

The ocean-atmosphere system is forced by the solar radiation. This forcing is usually very constant. Yet, observations show substantial fluctuations in both the atmosphere and the ocean at various time scales, such as the atmospheric cyclones, storms, marine eddies and El Nino. This variability occurs because the mean state is physically unstable to infinitesimal disturbances.

The simplest example of instability is the gravitational instability of a ball on a supporting surface. In principle, the ball is in an equilibrium state as long as the local supporting surface is flat. In practice, however, only the equilibrium state is physically achievable only when the nearby supporting surface is a well. In the case when the supporting surface concaves upwards, the ball will practically not stay in equilibrium because any infinitesimal perturbation will push the ball off the equilibrium position.

The simplest example of fluid instability is the convective instability in a stratified fluid. Given a density field as $\rho = \rho_0 + \rho_s(z) + \rho'(t,z)$ where $\rho_0 \gg \rho_s(z) \gg \rho'(t,z)$, the linearized vertical momentum and density equations are:

$$\rho_0 \partial_t w = -g\rho', \qquad \partial_t \rho' + w\partial_z \rho_s(z) = 0$$

That is

$$\partial_{tt} w = -g\partial_t \rho'/\rho_0 = -N^2 w$$

where $N^2 = -g\partial_z \rho_s/\rho_0$. If $N^2 > 0$, we have

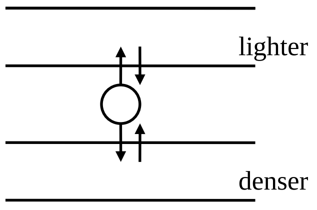
$$w = \exp(\pm iNt)$$

The solution is stable and oscillating at the initial amplitude, with N being the Brunt-Vasara frequency. If, however, $N^2 < 0$, we have

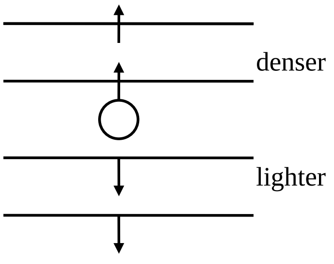
$$w = \exp(\pm|N|t)$$

The solution becomes unstable and the amplitude increases exponentially, with $|N|$ being the growth rate. This is the convective instability.

Intuitively, the convective instability is obvious. When $N^2 < 0$, the density of the background fluid increases upward. A fluid parcel displaced upward (downward) will find itself lighter (heavier) than the environment fluid and therefore will continue to rise (descend). From the energy viewpoint, the instability occurs because the center of gravity is lowered for the fluid such that its available potential energy is released to provide the kinetic energy.



Stable, $N^2 > 0$

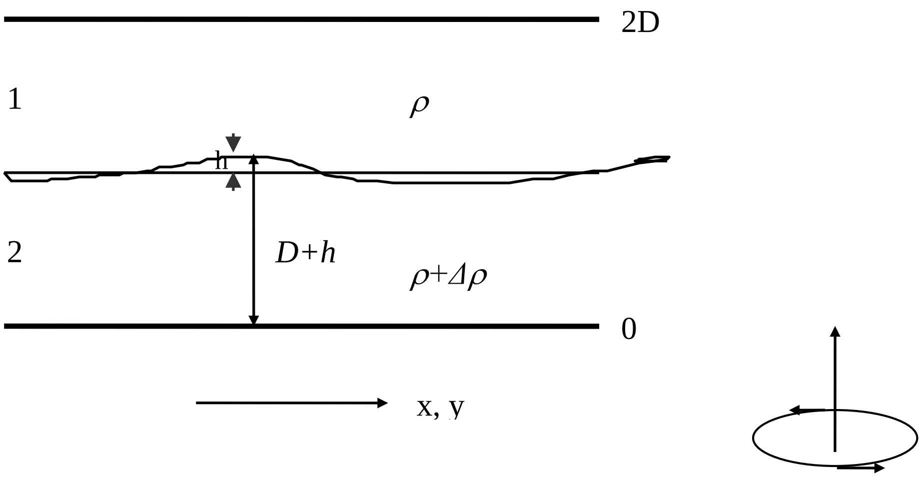


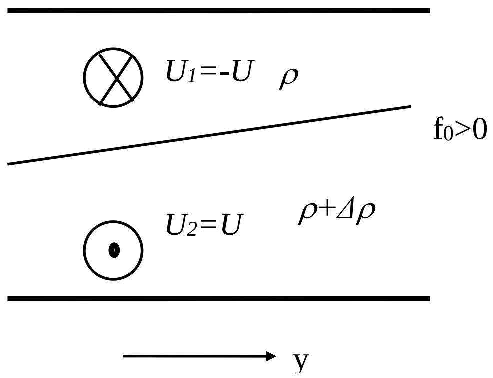
Unstable, $N^2 < 0$

Sec. 6.2: Baroclinic Instability: Phillip's Two-Layer Model.

1. The Two-Layer Model

We first study baroclinic waves in the simplest system that admits baroclinic instability -- the two-layer fluid. The system has two shallow-water layers of densities ρ and $\rho+\Delta\rho$ with $\Delta\rho<<\rho$. For simplicity, the fluid has a rigid lid and a flat bottom at $z=2D$ and $z=0$, respectively; the two layers have an equal depth D and are on a beta-plane.





If h is the departure of the interface depth from its undisturbed position, the potential vorticity under QG approximation in each layer is:

$$q_1 = \frac{f+\zeta_1}{D-h} = \frac{f_o+\beta y+\zeta_1}{D(1-h/D)} \approx \frac{f_o+\beta y+\zeta_1}{D}(1+\frac{h}{D}) \approx \frac{1}{D}(f_o+\beta y+\zeta_1+f_o\frac{h}{D}),$$

$$q_2 = \frac{f+\zeta_2}{D+h} = \frac{f_o+\beta y+\zeta_2}{D(1+h/D)} \approx \frac{f_o+\beta y+\zeta_2}{D}(1-\frac{h}{D}) \approx \frac{1}{D}(f_o+\beta y+\zeta_2-f_o\frac{h}{D}).$$

where $\zeta_n = \nabla^2\psi_n$ is the relative vorticity of layer n ($=1, 2$).

The inviscid shallow water QGPV equations are:

$$\partial_t q_n + J(\psi_n, q_n) = 0. \qquad (n=1, 2) \qquad (6.2.1)$$

where q_n and ψ_n are the QGPV and geostrophic streamfunctions of the two layers. In order to close the model, we need to express h in terms of ψ_n. At the surface, let $p(x, y, 2D, t) = p_T(x, y, t)$. The hydrostatic relation of layer 1 is $\partial_z p_1 = -g\rho$, which gives:

$$p_1(x, y, z, t) = p_T + g\rho(2D - z) \ , \qquad (6.2.2)$$

The geostrophic streamfunction of layer 1 is therefore

$$\psi_1 = p_T/\rho f_0 \ . \qquad (6.2.3)$$

The pressure at the interface $z=D+h$ is therefore $p_1|_{z=D+h} = p_T + g\rho(D-h)$. In the bottom layer, $\partial_z p_2 = -g(\rho+\Delta\rho)$. The continuity of pressure across the interface requires that $p_1|_{z=D+h} = p_2|_{z=D+h}$. Therefore,

$$p_2(x, y, z, t) = p_1|_{z=D+h} + g(\rho + \Delta\rho)(D + h - z) \ ,$$

$$p_2(x, y, z, t) = p_T + g\rho(2D - z) + g\Delta\rho(D + h - z) \ ;$$

The geostrophic streamfunction of layer 2 is:

$$\psi_2 = (p_T + gh\Delta\rho)/\rho f_0 \ . \qquad (6.2.4)$$

Therefore, the interface is related to the difference of streamfunction as:

$$\psi_1 - \psi_2 = -gh\Delta\rho/\rho f_0 = -g'h/f_0 \qquad (6.2.5)$$

where $g' = -g\Delta\rho/\rho$ is the reduced gravity. The difference of velocities is

$$(u_1 - u_2, \ v_1 - v_2) = (\partial_y, -\partial_x)g'h/f_0$$

This is the "thermal wind" relation in the two-layer model, relating the vertical shears to the horizontal gradients of the interface height (and therefore to the horizontal gradients of density along the mean position of the interface).

With (6.2.5), the QGPVs can be written in terms of ψ_n :

$$q_n = f_0 + \beta y + \nabla^2\psi_n + (-1)^n F(\psi_1 - \psi_2) \qquad (6.2.6)$$

where $F = 1/L_{D1}^2$ and $L_{D1} = (g'D)^{1/2}/f_0$ is the deformation radius of the first baroclinic mode -- the only baroclinic mode in the two-layer system. Eqns. (6.2.1) and (6.2.6) form a closed set of equations for ψ_1 and ψ_2.

2. Baroclinic Rossby Waves

In the absence of mean flows, the perturbation equations are:

$$\partial_t[\nabla^2\psi_1 - F(\psi_1 - \psi_2)] + \beta\partial_x\psi_1 = 0$$

$$\partial_t[\nabla^2\psi_2 + F(\psi_1 - \psi_2)] + \beta\partial_x\psi_2 = 0. \qquad (6.2.7)$$

Define the barotropic and baroclinic streamfunctions as:

$$\psi_B = \psi_1 + \psi_2 \ , \ \psi_C = \psi_1 - \psi_2 \ \propto -h \ ,$$

The summation and subtraction of the two perturbations equations give:

$$\partial_t\nabla^2\psi_B + \beta\partial_x\psi_B = 0 \ ,$$

$$\partial_t[\nabla^2\psi_C - 2F\psi_C] + \beta\partial_x\psi_C = 0 \ .$$

The barotropic equation is similar to the shallow water QG with the deformation radius set to infinity, and the baroclinic equation is the shallow water equation with the deformation radius set as the baroclinic deformation radius of the first mode: $L_{D1}^2 = 1/F$. Assuming the eigenfunctions of the form:

$$\psi_B = A_B e^{i(kx+ly-\omega t)}\ , \qquad \psi_C = A_C e^{i(kx+ly-\omega t)}$$

We have the eigenvalue problem form in (6.2.7) as:

$$(\omega K^2 + \beta k)A_B = 0 \quad (6.2.8a)$$

$$[\omega(K^2 + 2F) + \beta k]A_C = 0 \quad (6.2.8b)$$

where $K^2 = k^2 + l^2$ is the total wave number. The eigenvalues for the barotropic and baroclinic modes are therefore:

$$\omega_B = -\beta k/K^2 \quad (6.2.9a)$$

$$\omega_C = -\beta k/(K^2 + 2F) \quad (6.2.9b)$$

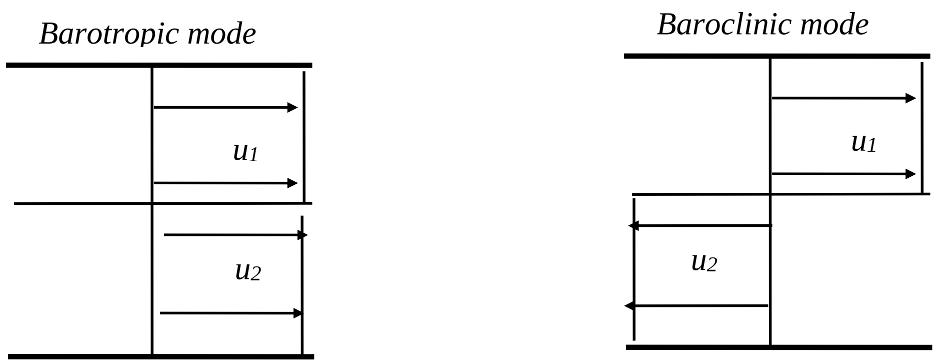
Substitute (6.2.9a) into (6.2.8b), we have the eigenfunction:

$$A_C = 0\ , \quad (6.2.10a)$$

or $\psi_1 - \psi_2 = 0$. Therefore, there is no vertical shear for the barotropic mode. In contrast, for the baroclinic mode, plug in (6.2.9b) into (6.2.8a), we have the eigenfunction:

$$A_B = 0\ , \quad (6.2.10b)$$

or $\psi_1 + \psi_2 = 0$. The baroclinic mode has no vertically integrated transport. These eigenfunction structures are consistent with the continuously stratified case in Section 5.3.



3. Baroclinic Instability

In the presence of mean flow shear, the eigenvalue problem is complicated dramatically. In particular, we may have unstable waves. Consider a basic state of equal and opposite zonal flows in the two layers $(U_1, V_1) = (U, 0)$ and $(U_2, V_2) = (-U, 0)$ where U is a constant. The basic state streamfunctions are: $\Psi_1 = -Uy$, $\Psi_2 = Uy$ and the interface shape is $H = -Sy$ where $S = 2f_0U/g'$. Separate the streamfunction into the mean and perturbation parts:

$$\psi_n = \Psi_n + \psi'_n$$

the linearized perturbation QGPV equations can be derived from (6.2.1) as:

$$(\partial_t + U\partial_x)q'_1 + \partial_y Q_1 \partial_x \psi'_1 = 0\ ,$$

$$(\partial_t - U\partial_x)q'_2 + \partial_y Q_2 \partial_x \psi'_2 = 0 \ .$$

Where $q'_n \ and \ Q_n$ are derived from (6.2.6) as the perturbation and mean PVs in layer *n*. Since the basic state PV gradients are:

$$[\partial_y Q_1,\ \partial_y Q_2] = [\beta + 2FU, \beta - 2FU] \ . \qquad (6.2.11)$$

We have:

$$(\partial_t + U\partial_x)\left[\nabla^2\psi_1 - F\left(\psi'_1 - \psi'_2\right)\right] + (\beta + 2FU)\partial_x\psi'_1 = 0$$

$$(\partial_t - U\partial_x)\left[\nabla^2\psi_2 + F\left(\psi'_1 - \psi'_2\right)\right] + (\beta - 2FU)\partial_x\psi'_2 = 0 \qquad (6.2.12)$$

Searching for the normal mode solutions of the form:

$$\psi'_n = Re[A_n e^{ik(x-ct)+ily}]$$

we have

$$[\beta + 2FU - (K^2 + F)(U - c]A_1 + F(U - c)A_2 = 0$$

$$F(U + c)A_1 + [2FU - \beta - (K^2 + F)(U + c]A_2 = 0 \ . \qquad (6.2.13)$$

These equations have non-trivial solution if the determinant of the coefficients vanishes, leading to the eigenvalues of

$$c = \{-\beta(K^2 + F) \pm [\beta^2F^2 - K^4U^2(4F^2 - K^4)]^{\frac{1}{2}}\}/[K^2(K^2 + 2F)] \qquad (6.2.14)$$

One can show that the case of $U=0$ recovers the two solutions (6.2.8a,b). In particular, both modes are neutral modes and there is no instability. However, when $U\neq 0$, instability may occur.

Case I: $\beta=0$

In the simpler case of $\beta=0$, the eigenvalues in (6.2.14) become:

$$c = \pm[-K^4U^2(4F^2 - K^4)]^{\frac{1}{2}}\}/[K^2(K^2 + 2F)] \qquad (6.2.15)$$

Thus, instability occurs with $Im(c) = c_i > 0$ when the wave number satisfies:

$$K^2 < 2F \qquad (6.2.16)$$

The solution grows exponentially with time as $e^{ik(-ic_it)} = e^{kc_it}$, and initial disturbances of infinitesimal amplitude amplifies rapidly. The instability condition (6.2.16) indicates that baroclinic instability has a short wave cut off, that is the instability does not occur for short waves of $K^{-1} < 0.707L_{D1}$.

Case II: General case: $U\neq 0, \beta\neq 0$

In the general case, in addition to (6.2.16), unstable waves must also satisfy

$$U^2 > \beta^2F^2/[K^4(4F^2 - K^4)] \equiv U_c^2 \ . \qquad (6.2.17)$$

The β effect is therefore a stabilizing effect, because it imposes a critical shear U_c. This critical shear U_c is a function of $K^2/2F$, as shown in the following figure. Furthermore, for $K^2 < 2F$, U_c^2 has the minimum value of $\beta^2/(2F)^2 \ at \ K^2 = 2^{1/2}F$. The unstable modes occur only for

$$|U| > \beta/2F \ . \qquad (6.2.18)$$

This is precisely the condition that the mean PV gradients in (6.2.11) have the opposite signs in the two layers:

$$\partial_y Q_1 = \beta + 2FU > 0, \quad \partial_y Q_2 = \beta - 2FU < 0, \text{ for eastward shear flow } U > \beta/2F > 0$$

Or

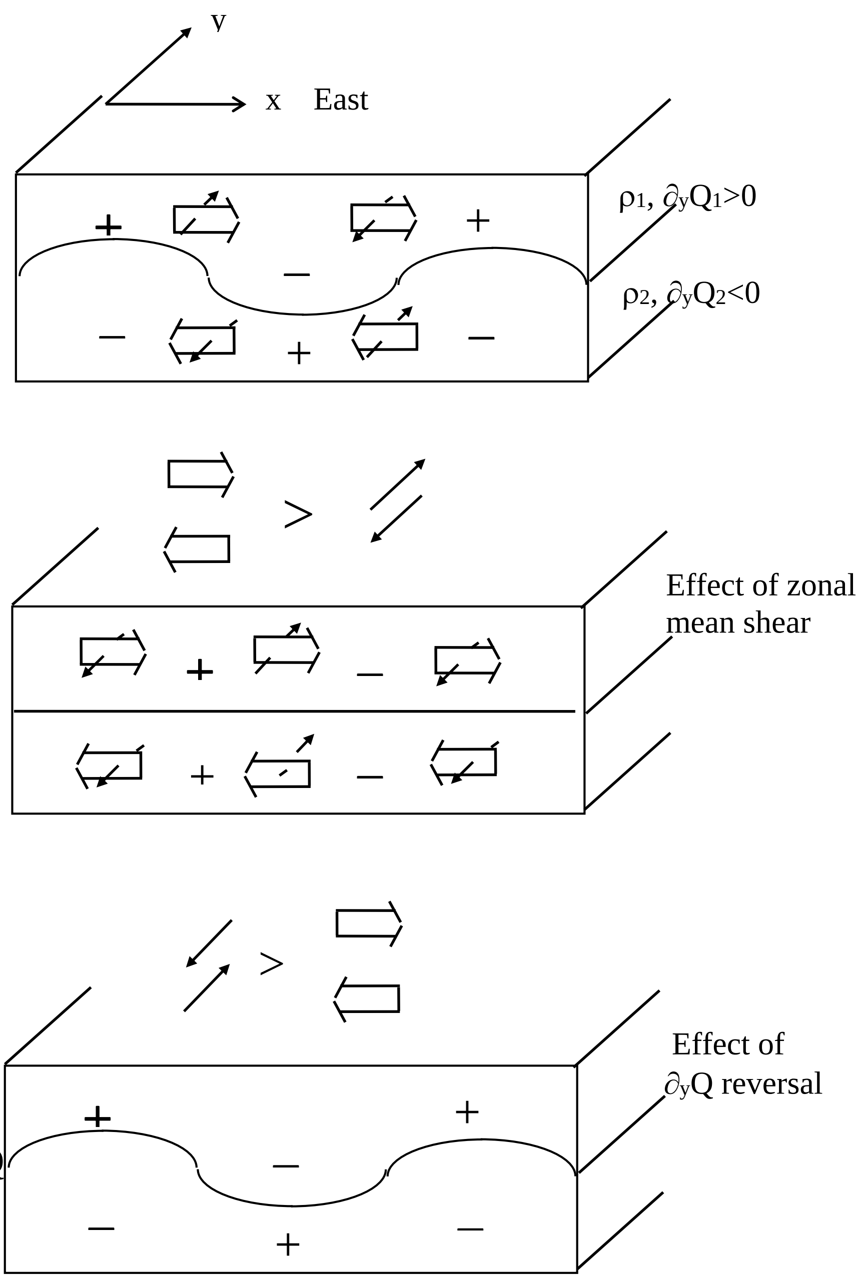
$$\partial_y Q_1 = \beta + 2FU < 0, \quad \partial_y Q_2 = \beta - 2FU > 0, \text{ for westward shear flow } U < -\beta/2F < 0$$

This reversal of PV gradients can be shown as a necessary condition for instability (Charney-Stern theorem, Section 6.4). Here, in the 2-layer model, it is also the sufficient condition. The second figure shows the real and imaginary parts of c when U is twice this critical value (see Pedlosky, Fig.7.11.2).

3. Mechanism of Instability

The baroclinic instability can be interpreted using the vorticity argument of Bretherton (1966). The instability can be thought to occur in two steps. Start with an initial baroclinic vorticity disturbance in panel (a). First, the mean shear flow converts the baroclinic vorticity component to the barotropic vorticity component, as shown in panel (b). Here, the mean shear is necessary to for the coupling of the barotropic and baroclinic modes. Second, the perturbation meridional velocity of the barotropic component, due to opposite sign of mean PV gradients, produces opposite (or baroclinic) vorticity tendencies in the two layers. This reinforces the initial baroclinic anomaly, forming a positive feedback or baroclinic instability. It is seen that both the mean shear and the reversal of the mean PV gradient are critical for baroclinic instability. Later, the role of the mean shear can also be seen in Section 6.3 from the energy conversion viewpoint, while the role of the reversed PV gradient can be seen in Section 6.4 in the Charney-Stearn theorem. The shortwave cut-off can be understood as follows: due to the finite thickness of each layer, very short waves can't feel the reversal of the mean PV gradients and therefore are stable.

Finally, the coupling between the baroclinic and barotropic modes (which is possible only with mean shear) is critical, as seen in the dispersion diagram (Fig.6.1). Instability occurs when both modes have the same speed. This also explains the absence of very long wave baroclinic instability in the two-layer model, or the long wave cut off. In the long wave limit, the barotropic wave is infinitely fast while the baroclinic wave remains finite. It is impossible for the two modes to have comparable speed and therefore the two modes can't couple with each. (In the continuously stratified case in Section 6.5, the long wave cut off disappears).



a. ζ

b. $\partial_t \zeta \sim -U\partial$

c. $\partial_t \zeta \sim -v' \partial_y Q$

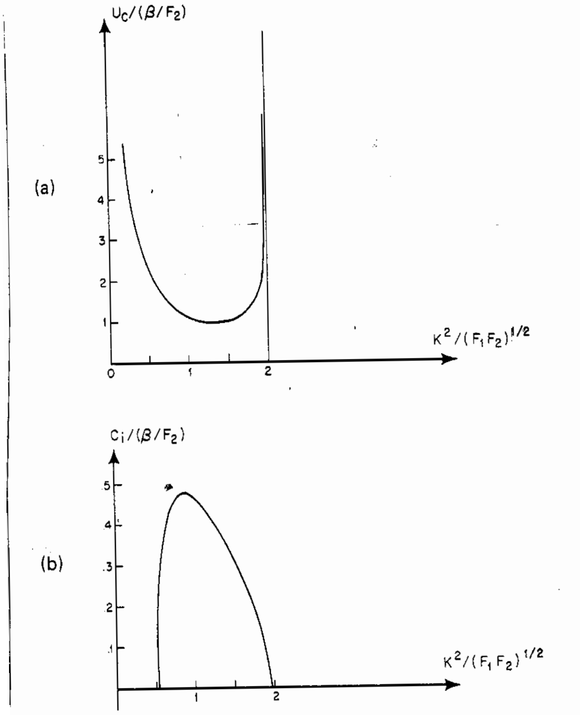


Figure 7.11.1 (a) The critical shear U_c as a function of wave number in the case $F_1 = F_2$. For $U_s > U_c$ *and* $K^2 < 2(F_1F_2)^{1/2}$, c is complex. Elsewhere c is strictly real. (b) The imaginary part of c as a function of wave number for $U_s = 2\beta/F_2$, i.e., for a shear which is twice the minimum critical shear.

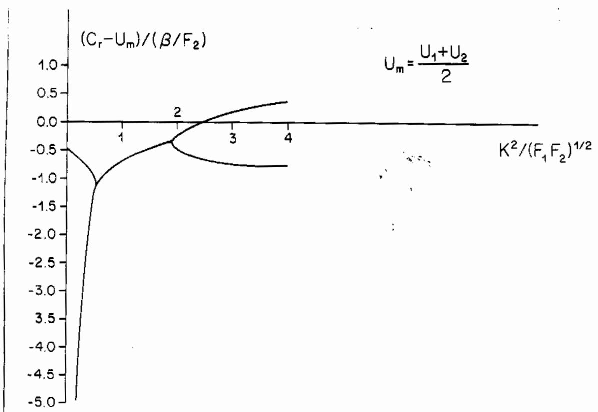


Figure 7.11.2 The real part of c as a function of wave number when $U_s = 2\beta/F_2$.

Fig.6.1: Eigenvalues of the 2-layer baroclinic model

Sec. 6.3: Energetics

The instability can also be understood from the energy exchange between the wave and the mean flow. We will derive the energy equations for the wave and the mean.

1. Total QG Energy Equation

In the absence of forcing and dissipation, the QGPV $q = \beta y + \nabla^2\psi + \left(\frac{1}{p}\right)\partial_z\left(\frac{pf_0^2}{N^2\psi_z}\right)$ satisfies the equation:

$$\partial_t q + J(\psi, q) = 0\ . \tag{6.3.1}$$

The energy equation can be derived similar to the shallow water case in section 2.1. Multiply (6.3.1) by -ψ, and integrate the equation, we have:

$$\iiint -\psi[\partial_t q + J(\psi, q)] = 0\ .$$

Notice

$$-\psi\partial_t\partial_z[pf_0^2/N^2\partial_z\psi] = -\psi\partial_z[pf_0^2/N^2\partial_{tz}\psi] = \partial_z[-\psi pf_0^2/N^2\partial_{tz}\psi] + \partial_z\psi pf_0^2/N^2\partial_{tz}\psi$$
$$= \partial_z[-\psi pf_0^2/N^2\partial_{tz}\psi] + \partial_t[(\partial_z\psi)^2 pf_0^2/2N^2],$$

$$-\psi\partial_t\psi_{xx} = \partial_x[\psi\partial_{tx}\psi] + \partial_t[(\partial_x\psi)^2/2],$$

$$-\psi\partial_t\psi_{yy} = \partial_y\left[\psi\partial_{ty}\psi\right] + \partial_t[(\partial_y\psi)^2/2],$$

We have

$$\iiint \partial_t\left\{(p/2)[(\partial_x\psi)^2 + (\partial_y\psi)^2 + \frac{pf_0^2}{N^2}(\partial_z\psi)^2]\right\}$$
$$= \iiint p\psi J(\psi, q) + \iint dxdz\, p\psi\partial_{ty}\psi\,|_{y_1}^{y_2} + \iint dxdy\, p\frac{f_0^2}{N^2}\psi\partial_{tz}\psi\,|_{z_1}^{z_2}$$

If the boundary conditions are periodic or solid wall in *x*, *y*, the first two terms on the RHS vanish. The last term also vanishes because we have from the hydrostatic equation $\theta \sim \partial_z\psi$. On the top and bottom boundaries *w=0*, we have from the thermodynamic equation $-\psi[\partial_t\theta + J(\psi,\theta)] = 0$, and therefore

$$\iint dxdy\psi\partial_{tz}\psi \propto \iint dxdy\psi\partial_t\theta = \iint dxdy\psi J(\psi,\theta) = \iint dxdy J(\psi^2/2, \theta) = 0,$$

Thus, we have the total energy conservation:

$$\partial_t(KE + APE) = 0, \tag{6.3.2}$$

where the total kinetic energy and available potential energy are:

$$KE = \iiint p\left[(\partial_x\psi)^2 + (\partial_y\psi)^2\right]/2, \quad \mathrm{AP}E = \iiint p(\partial_z\psi)^2 f_0^2/2N^2 \tag{6.3.3}$$

2. Mean and Perturbation Energy Equations:

The QGPV equation. (6.3.1) can be rewritten as:

$$\partial_t q + \nabla\cdot(\boldsymbol{u_g} q) = 0, \tag{6.3.4}$$

because $\nabla \bullet \boldsymbol{u}_g = 0$. Now, we derive the energy equation for the zonal mean flow. We will denote a variable as $a = A + a'$ where $A=\langle a\rangle$ is the zonal average and a' is the perturbation. Notice the definition of the E-P flux in (5.4.3), the zonal mean QGPV equation is:

$$\partial_t Q = -\langle \nabla \cdot (\boldsymbol{u_g} q)\rangle = -\partial_y(\langle \boldsymbol{u_g}\rangle\langle q\rangle) - \partial_y(\langle u'_g q'\rangle)$$
$$= -\partial_y(\langle \boldsymbol{u_g}\rangle\langle q\rangle) - \partial_y\langle \nabla \cdot \boldsymbol{F}\rangle$$

Multiplying the equation by $-\Psi$ and integrate it as we did for the total energy equation (6.3.2), we have:

$$\partial_t(K_m + A_m) = \iiint \Psi \partial_y \nabla \cdot \boldsymbol{F} = \iiint U \nabla \cdot \boldsymbol{F} \, . \tag{6.3.5}$$

where the kinetic and available potential energy for the mean flow are**:**

$$K_m = \iiint p(\partial_y \Psi)^2/2, \qquad A_m = \iiint p(\partial_z \Psi)^2 f_0^2/2N^2, \tag{6.3.6}$$

Notice that $\langle \psi^2\rangle = \langle (\Psi + \psi')^2\rangle = \langle \Psi^2\rangle + \langle \psi'^2\rangle$, the total KE and APE are the sum of the mean and perturbation, i.e.

$$KE = K_m + K', \quad APE = A_m + A' \tag{6.3.7}$$

where the perturbation energy is

$$K' = \iiint p\left[(\partial_x \psi')^2 + (\partial_y \psi')^2\right]/2, \quad \mathrm{A}' = \iiint p(\partial_z \psi')^2 f_0^2/2N^2 \tag{6.3.8}$$

The conservation of the total energy and the energy for the mean flow gives the energy equation for the perturbation flow as

$$\partial_t(K' + A') = -\iiint U \nabla \cdot \boldsymbol{F} = -\partial_t(K_m + A_m) \, . \tag{6.3.9}$$

Thus, the convergence of the E-P flux also represents the conversion of energy between the perturbation and the mean flow. The conversion term can be rewritten as:

$$\partial_t(K' + A') = -\iiint U(y,z,t) \nabla \cdot \boldsymbol{F} = -\iint dydz U \, \nabla \cdot \boldsymbol{F}$$
$$= -\iint dydz U \partial_y F_y - \iint dydz U \partial_z F_z$$
$$= \iint dydz F_y \partial_y U + \iint dydz F_z \partial_z U \tag{6.3.10}$$

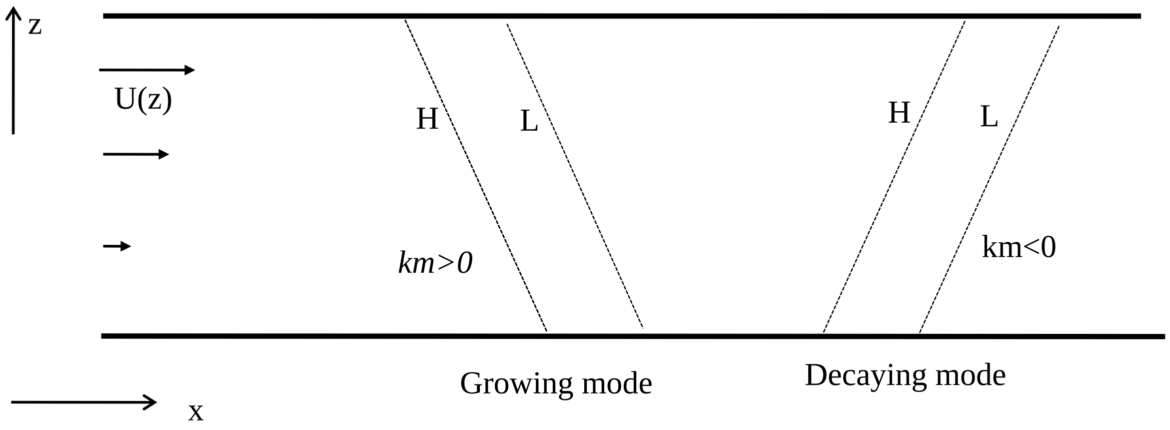
The last step has used the E-P flux vanishes on the y and z boundaries. On the RHS, the first term is the barotropic conversion term, which involves the horizontal shear of the mean flow and the conversion of the mean and the perturbation kinetic energy; the second term is the baroclinic conversion term, which involves the vertical shear (horizontal temperature gradient) and the conversion of the mean and perturbation APE.

First, the shear of the mean flow is necessary for the energy conversion and in turn instability. The instability associated with the horizontal and vertical shear is called the barotropic and baroclinic instability, respectively.

Second, the structure of the perturbation is also important for instability. For example, for baroclinic instability, in a westerly shear flow $\partial_z U > 0$ in the midlatitude (or $\partial_y T < 0$), the unstable waves need to have a upward E-P flux:

$$F_z \sim \langle v'\theta'\rangle \; > \; 0,$$

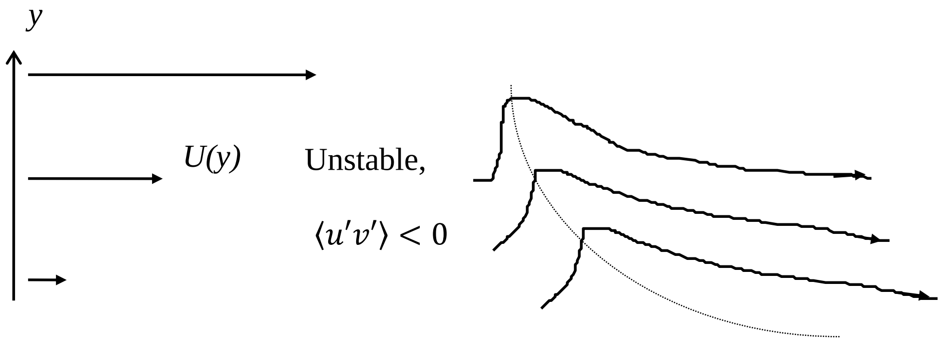
Therefore, the unstable waves transport heat towards the pole, reducing the mean temperature gradient and releasing the mean APE to the perturbation APE. The unstable wave also propagates upward and tilts westward with height ($\langle v'\theta'\rangle \propto \langle \partial_x \psi' \partial_z \psi'\rangle \;\; \propto km|\Psi|^2$).



In the case of barotropic instability, for a mean flow of $\partial_y U > 0$, the unstable disturbance will have

$$F_y \sim -\langle u'v'\rangle > 0,$$

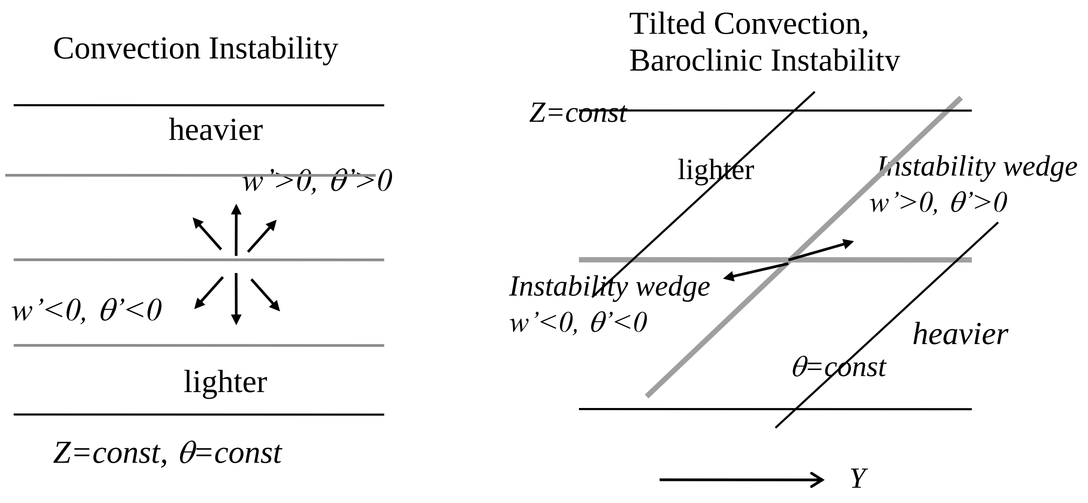
Now, the energy is released from the mean kinetic energy to the perturbation kinetic energy.



The study of the energetics gives necessary conditions of instability and is valid for any type of small disturbance. It however should be realized that, this approach, although powerful and insightful, does not provide sufficient conditions for instability.

3. Energetics of Baroclinic Instability

The energetics of baroclinic instability can also be understood from parcel displacements. First, take the convective instability as an example. For a perturbation parcel to gain kinetic energy, the mean state has to release the mean APE. This can be achieved if the parcels transport heat upward $\langle w'\theta'\rangle > 0$, which warms up upper layers and therefore lowers the center for gravity. In a density profile that increases upward (or $\partial_z\Theta < 0$), any parcel trajectory satisfy the upward heat transport $\langle w'\theta'\rangle > 0$. This is because an upward parcel $w' > 0$ is warmer than its environment and therefore has $\theta' > 0$, while a downward parcel $w' < 0$ is cooler than its environment and therefore has $\theta' < 0$. On the other hand, if the density increases downward, no parcel trajectory produces an upward heat transport. Therefore, the convective instability is independent of the structure of the perturbation (as seen in section 6.1). This will not be the case for baroclinic instability.



Assume the mean stratification is convectively stable ($\partial_z\Theta > 0$) for parcels moving in the vertical direction, but the mean isothermal slopes with latitude ($\partial_y\Theta < 0$). Now, the parcel must lie within the "instability wedge" (see the figure above) to transport heat upward $w'\theta' > 0$. In other words, the parcel must travel in tilted trajectories. Therefore, baroclinic instability is also called tilted convection. In the mean time, the tilted convection also has $v' > 0$ in the wedge of $w' > 0, \theta' > 0$ (and $v' < 0$ in the wedge of $w' < 0, \theta' < 0$). The instability wedge $w'\theta' > 0$ also produces northward heat flux $v'\theta' > 0$, reducing the mean equatorward temperature gradient and in turn the mean APE.

These energy conversions from the mean APE to perturbation APE to the perturbation KE can be seen explicitly if we derive the equation for the perturbation APE. The perturbation thermodynamic equation is:

$$(\partial_t + U\partial_x)\theta' + v'\partial_y\Theta + w'd\Theta/dz = 0\ ,$$

Or

$$(\partial_t + U\partial_x)\theta' = -\mathbf{v}' \cdot \nabla\Theta\ ,$$

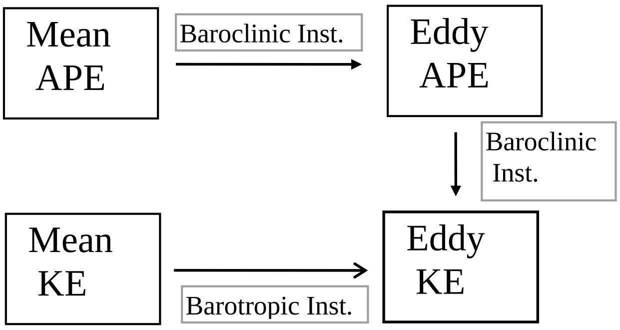
Multiply the equation by $pg\theta'/(\Theta_s d\Theta/dz)$, and integrate the equation lead to the equation for the wave APE as

$$\partial_t A' \equiv \partial_t \iiint pg\theta'^2/(2\Theta_s d\,\Theta/dz) = -\iiint pg/(\Theta_s d\,\Theta/dz)\mathbf{v}'\theta' \cdot \nabla\Theta$$
$$= -\iiint v'\theta'\partial_y\Theta pg/(\Theta_s d\,\Theta/dz) - \iiint w'\theta'\partial_z\Theta_s pg/(\Theta_s d\,\Theta/dz)$$

The first term converts the mean APE to the perturbation APE, while the second term converts the perturbation APE to perturbation KE. Therefore, in the instability wedge when $\mathbf{v}'\theta' \cdot \nabla\Theta < 0$, there is an energy release from mean APE to the perturbation APE and finally the perturbation KE. This is the energy cycle of the baroclinic waves:

$$A_m \Longrightarrow A' \Longrightarrow K'$$

Energy Cycle:

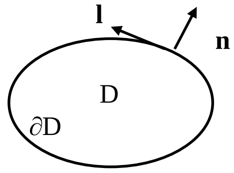


Sec. 6.4: Charney-Stearn Theorem

1. Charney-Stearn Theorem:

Complementary to the study of the energetics in the last section, Charney-Stearn theorem studies the instability in terms of the mean potential vorticity field. Consider small amplitude perturbations on a basic state of parallel flow $\Psi(y,z)$ and of a buoyancy frequency $N(z)$, the mean PV field is:

$$Q(y,z) = \beta y + \Psi_{yy} + 1/p\partial_z[pf_0^2/N^2\partial_z\Psi]$$



For small amplitude perturbations, the wave activity equation is

$$\partial_t A + \nabla \cdot \mathbf{F} = 0$$

where $A = p\langle q'^2 \rangle / 2Q_y$, with ⟨ ⟩ being the zonal mean. Integrate the equation over a domain D bounded in the y, z plane by ∂D, we have:

$$\partial_t \iint_D A + \int_{\partial D} \mathbf{F} \cdot \mathbf{n} dl = 0 \quad (6.4.1)$$

where $\boldsymbol{n}$ is the outward unit normal to ∂D. If ∂D is a rigid boundary, one might expect **F•n** to vanish there. That is, however, not the case.

For simplicity, we take D as the region of $y_1 \le y \le y_2$ and $z_1 \le z \le z_2$. Immediately, we have v'=0 on the y boundaries

$$\mathbf{F} \cdot \mathbf{n} = F_y = -p\langle u'v' \rangle = 0 \quad y = y_1,\ y_2 \quad (6.4.2)$$

However, on the upper and lower boundaries, the situation is subtle, because the vertical component of E-P flux is not vertical velocity. On the boundary surface, we have $w=0$, and therefore the thermodynamic equation:

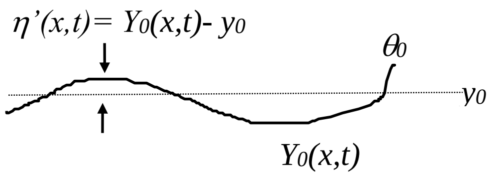
$$\partial_t \theta + J(\psi, \theta) = 0, \qquad z = z_1,\ z_2 \quad (6.4.3)$$

The linearized equation is

$$(\partial_t + U\partial_x)\theta' + v'\Theta_y = 0, \qquad z = z_1,\ z_2 \quad (6.4.4)$$

We define a northward displacement η' of the wave motion for a given isothermal θ_0 that is defined by $\theta_0 \equiv \theta[x, Y_0(x,t), t]$. The kinematic condition $dY_0(x,t)/t = v$ can be linearized as

$$(\partial_t + U\partial_x)\eta' = v'.$$



Substitute this into (6.4.4), we have:

$$(\partial_t + U\partial_x)\theta' + \Theta_y(\partial_t + U\partial_x)\eta' = 0, \qquad z = z_1,\ z_2$$

Thus,

$$\theta' = -\eta'\Theta_y \qquad z = z_1,\ z_2$$

if we assume $\theta' = 0$ for $\eta' = 0$ (back in the far past before the disturbance was present). Therefore, the heat transport on the z boundaries is

$$\langle v'\theta'\rangle = -\langle v'\eta'\rangle\Theta_y = -\partial_t\langle \eta'^2\rangle\Theta_y/2, \qquad z = z_1,\ z_2 \qquad (6.4.5).$$

$$F_z = pf_0\langle v'\theta'\rangle/\Theta_z = -pf_0\partial_t\langle\eta'^2\rangle\Theta_y/2\Theta_z$$

Therefore, the normal component of the E-P flux $F_z = pf_0\langle v'\theta'\rangle/\Theta_z$ is zero for a growing disturbance $\partial_t\langle\eta'^2\rangle \neq 0$ only if the mean temperature gradient is zero there $\Theta_y = 0$. Otherwise, F_z has the opposite sign to $f_0\Theta_y$. On the surface, since Θ_y decreases poleward, F_z is always upward for growing disturbances. For almost-plane waves, one can easily show that the trough of the growing disturbance tilts westward with height.

If, however, there is no mean temperature gradient on the top and bottom boundaries, we have $F_z = 0$ on the boundaries. Therefore, $\int_{\partial D}\mathbf{F}\cdot\mathbf{n}dl = 0$ and we have the conservation of total wave activity from (6.4.1):

$$\partial_t \iint_D A\,dydz = 0 \qquad (6.4.6)$$

Since $A = p\langle q'^2\rangle/2Q_y$, we have:

$$\partial_t \iint_D p\langle q'^2\rangle/2Q_y\,dydz = 0\ .$$

For normal modes, q' can be written as:

$$q'(x,y,z,t) = r(t)q(x,y,z)\ , \qquad (6.4.7)$$

Therefore, we have

$$\partial_t r^2(\iint_D p\langle q^2\rangle/2Q_y)dydz = 0\ . \qquad (6.4.8)$$

If Q_y is single signed throughout, the integral constant cannot be zero, so

$$\partial_t r^2 = 0$$

The Charney-Stearn theorem therefore states: there can be no growing, conservative, quasi-geostrophic normal mode disturbances to a zonally-uniform state in which the potential vorticity gradient is single-singed throughout *and* which is isothermal on horizontal boundaries.

It is worth reflecting that the normal mode assumption (6.4.7), which grows or decays everywhere, is important for deriving (6.4.8). Otherwise, the disturbance can grow and decay at the same time within the domain. Therefore, the single signed PV gradient cannot guarantee the stability of the disturbance within the domain.

2. Necessary Condition for Instability

A sufficient condition for stability is also a necessary condition for instability. Thus, instability is possible if one of these conditions is violated. If the boundaries are isothermal, we require the change of sign of the mean PV gradient

$$Q_y = \beta - U_{yy} - 1/p\partial_z[pf_0^2/N^2U_z]$$

Since $\beta > 0$, Q_y must be positive unless the flow curvature - the latter two terms on the RHS - is negative and of sufficient magnitude, or

$$U_{yy} + 1/p\partial_z[pf_0^2/N^2U_z] > \beta \qquad \text{somewhere in the domain.} \tag{6.4.9}$$

The instability is called the barotropic instability if the first term (horizontal shear) is dominant, and is called the baroclinic instability if the second term (vertical shear, or horizontal temperature gradient) or the temperature gradient on the top and bottom boundaries is dominant.

3. The Boundary Temperature Effect*

The effect of the boundary temperature gradient can also be unified with the mean PV gradient by introducing a delta function on the boundary (Bretherton, 1966). We introduce the generalized QGPV as:

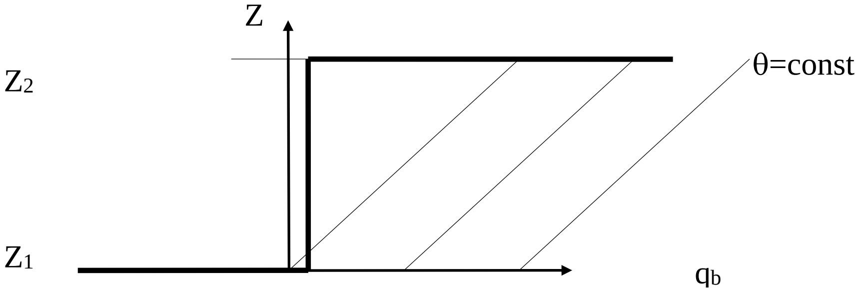
$$q_b = q + [\delta(z - z_1) - \delta(z - z_2)]f_0\theta/\, d\Theta/dz \tag{6.4.10}$$

This is like adding a PV sheet just inside a horizontal boundary. This generalized PV satisfies the QGPV equation:

$$\partial_t q_b + J(\psi,\, q_b) = 0,$$

and the boundary condition is now

$$\theta = 0 \qquad\qquad z = z_1,\, z_2 \tag{6.4.11}$$



This can be verified. In the interior, the delta functions are zero and (6.4.10) is simply

$$\partial_t q + J(\psi, q) = 0\ .$$

On the boundaries $z_1 +$ and $z_2 -$, since $\Theta = \Theta(z)$, (6.4.10) simply gives the boundary condition (6.4.3). The definition q_b thus incorporates both the interior and boundary equations. Since now $\theta' = 0$ on the top and bottom boundaries as in (6.4.11), $F_z = 0$ there. However, since $\nabla \cdot \boldsymbol{F} = \langle v'q'_b \rangle$**,** now we found in the interior

$$\nabla \cdot \boldsymbol{F} \approx \langle v'\theta' \rangle [\delta(z - z_1) - \delta(z - z_2)] f_0/\, d\Theta/dz\ ,$$

near z_1 and z_2. It must be the vertical component of **F** that is discontinuous within the boundary potential vorticity sheets and therefore integrated vertically in the vicinity of z_1 and z_2, we have:

$$F_z = \ \langle v'\theta' \rangle f_0/\, d\Theta/dz \qquad on\ z = z_1 +\ ,$$

$$F_z = -\langle v'\theta' \rangle f_0/\, d\Theta/dz \qquad on\ z = z_2 -\ .$$

The advantage of incorporate the boundary PV sheets to the generalized PV field is that we can follow the same steps as before, but with the boundary condition $F_z = 0$ on the vertical boundaries. The flow is stable if $\partial_y Q_b$ does not change its sign, or the flow is unstable only if the generalized mean PV gradient

$$\partial_y Q_b = \partial_y Q + [\delta(z - z_1) - \delta(z - z_2)] f_0 \partial_y \Theta/\, d\Theta/dz\ , \tag{6.4.12}$$

changes sign somewhere in the domain.

Sec. 6.5: Baroclinic Instability: Eady Model*

1. Eigenvalue Problem

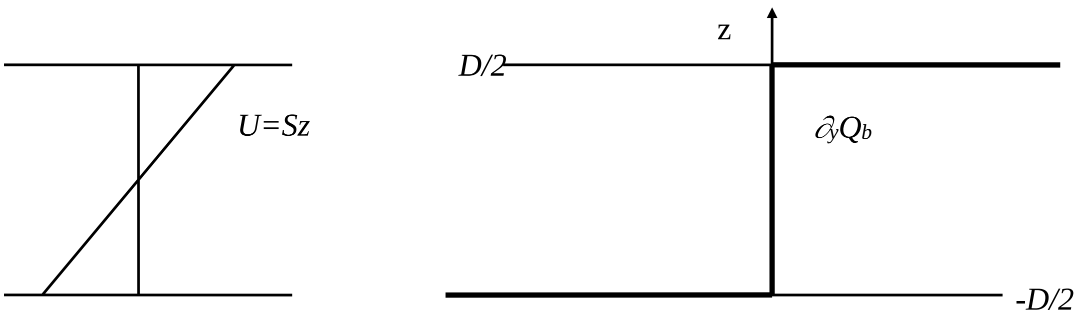
We consider the simplest example of baroclinic instability in a continuously stratified flow (Eady, 1949). The fluid is on an f-plane, unbounded in *x* and *y* directions, but bounded by rigid boundaries at $z = \pm D/2$, incompressible (so $p=p_0$, $H=\infty$, $\theta=T$). The basic state is a zonal flow $U = Sz$ ($\Psi = -Syz, T = -f_0 T_s Sy/g$) with a uniform stratification $N^2 = g\overline{T}_z/T_0$, giving a uniform mean PV in the interior, or

$$\partial_y Q = \partial_{yy} U + f_0^2/N^2 \partial_{zz} U = 0 \ . \tag{6.5.1}$$

There is, however, a potential for instability, since the mean temperature has gradients on $z = \pm D/2$. Consequently, the generalized mean PV Q_b has a gradient

$$\partial_y Q_b = [\delta(z - D/2) - \delta(z + D/2)] f_0 \partial_y T/\overline{T}_z \ , \tag{6.5.2}$$

negative on $z = -D/2$ but positive on $z = D/2$.



The linearized QGPV eqn is:

$$(\partial_t + U\partial_x) q' = 0 \tag{6.5.3}$$

where (if we insist $q' = 0$ for $t \to -\infty$)

$$q' = (\partial_{xx} + \partial_{yy})\psi' + f_0/N^2 \partial_{zz}\psi' \tag{6.5.4}$$

If we seek the solution of the form $\psi' = Re\{\psi e^{ik(x-ct)+ily} Z(z)\}$, the vertical structure is determined by:

$$d^2Z/dz^2 - \mu^2 Z = 0 \ , \tag{6.5.5}$$

where $\mu = NK/f_0,\ K^2 = k^2 + l^2$. This gives the general solution

$$Z = A sinh(\mu z) + B cosh(\mu z) \ . \tag{6.5.6}$$

The boundary conditions are:

$$(\partial_t + U\partial_x) T' + v' \partial_y T = 0, \qquad z = \pm D/2 \tag{6.5.7}$$

Since $T' = f_0 T_s \partial_z \psi'/g$ and $\partial_y T = -f_0 \partial_y T_s S/g$, the boundary conditions become

$$(Sz - c) dZ/dz - SZ = 0, \qquad z = \pm D/2 \ .$$

With (6.5.6), we have

$$A[\mu(c + \tfrac{SD}{2}) - S tanh(\tfrac{\mu D}{2})] + B[S - \mu(c + \tfrac{SD}{2}) tanh(\tfrac{\mu D}{2})] = 0, \quad z = -\tfrac{D}{2}\ , \tag{6.5.8a}$$

$$A\left[\mu\left(c-\frac{SD}{2}\right)+Stanh\left(\frac{\mu D}{2}\right)\right]+B\left[S+\mu\left(c-\frac{SD}{2}\right)tanh\left(\frac{\mu D}{2}\right)\right]=0, \quad z=\frac{D}{2} \qquad (6.5.8b)$$

Therefore, the eigenvalue is determined by setting the determinant of (6.5.8) zero as:

$$c^2 = (SD/\mu)^2\,[1+(\mu D/2)^2-\mu Dcoth(\mu D)] \qquad (6.5.9)$$

where we have used the identity $tanh\left(\frac{\mu D}{2}\right)+\left[\tanh(\frac{\mu D}{2})\right]^{-1} = 2coth(\mu D)$. The condition for instability is therefore

$$\mu Dcoth(\mu D)-(\mu D/2)^2 > 1\ , \qquad (6.5.10)$$

which implies

$$\mu D < d_c = 2.3994\ .$$

Since $\mu D = NKD/f_0 = L_{D1}/L$ and L_{D1} is the deformation radius for the first baroclinic mode, instability occurs when the scale of the wave is comparable or longer than L_{D1}.

2. Stable Rossby Waves

The dependency of the eigenvalue *c* and the eigenfunction structure *Z (z)* on μD are shown in Fig.6.1. For $\mu D > d_c$, the perturbation are two neutrally propagating modes. The structures of the stable waves (for $\mu D >> d_c$) are boundary trapped: the mode with negative *c* is trapped near the top ($A\approx 0$) while the mode with positive c is bottom trapped ($B\approx 0$). These two modes are boundary trapped Rossby waves. Although $\beta=0$ in the interior, the PV sheets on the boundaries provide the necessary PV gradient. The generation of the boundary trapped Rossby waves is very similar to the topographically generated Rossby waves in the stratified flow (Fig.6.2). The top boundary is like a northward shoaling bottom slope, and therefore acts as a positive beta with "topographic" Rossby waves propagating westward. The bottom boundary, however, is like a northward deepening bottom slope, and therefore acts like a negative beta, with "topographic" Rossby waves propagating eastward. Due to the boundary trapping, these waves don't feel the reversal of the mean PV gradient on the top and bottom boundaries simultaneously and therefore are stable.

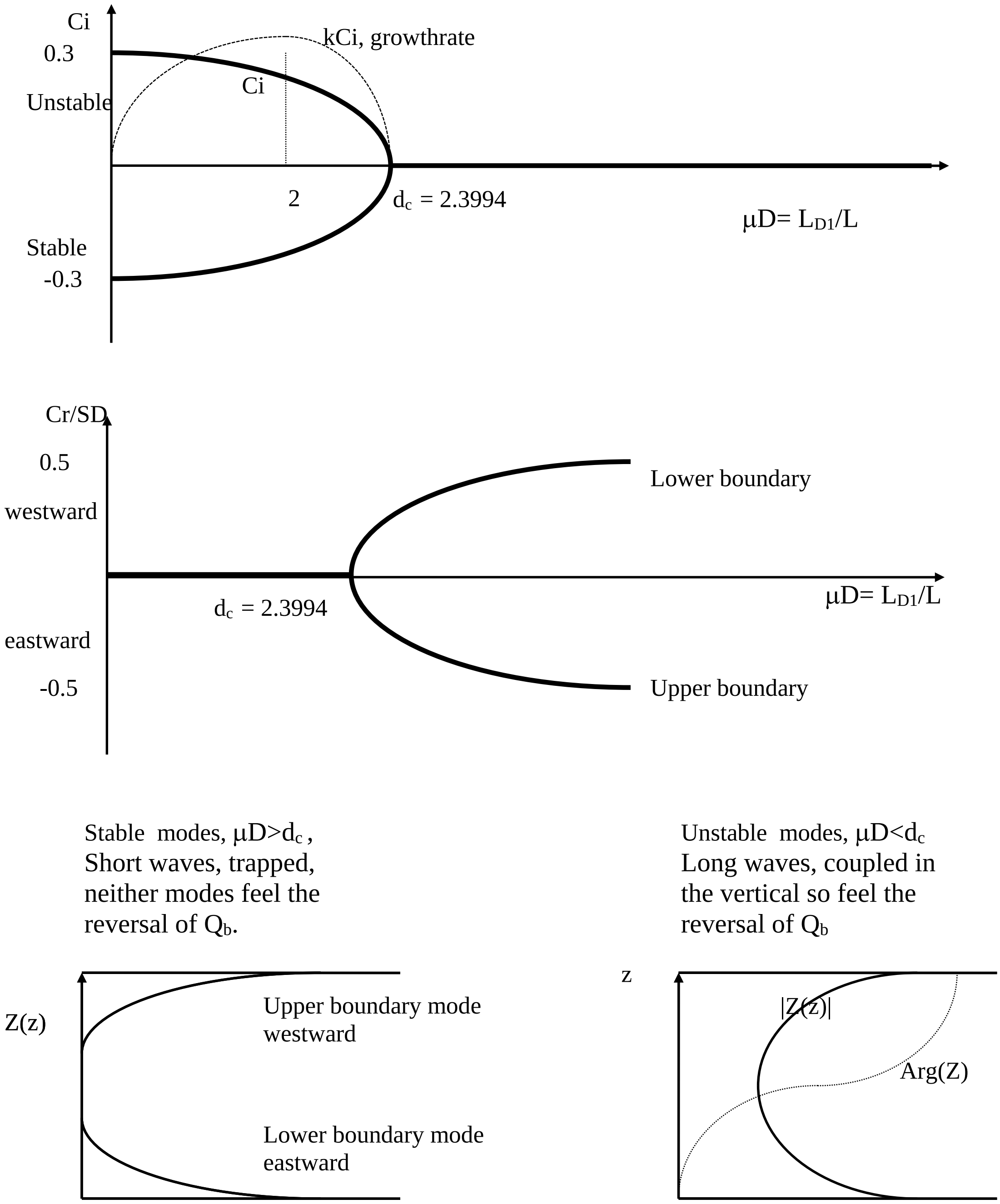


Fig.6.1 Eigenvalues and Eigenfunctions of the Eddy problem

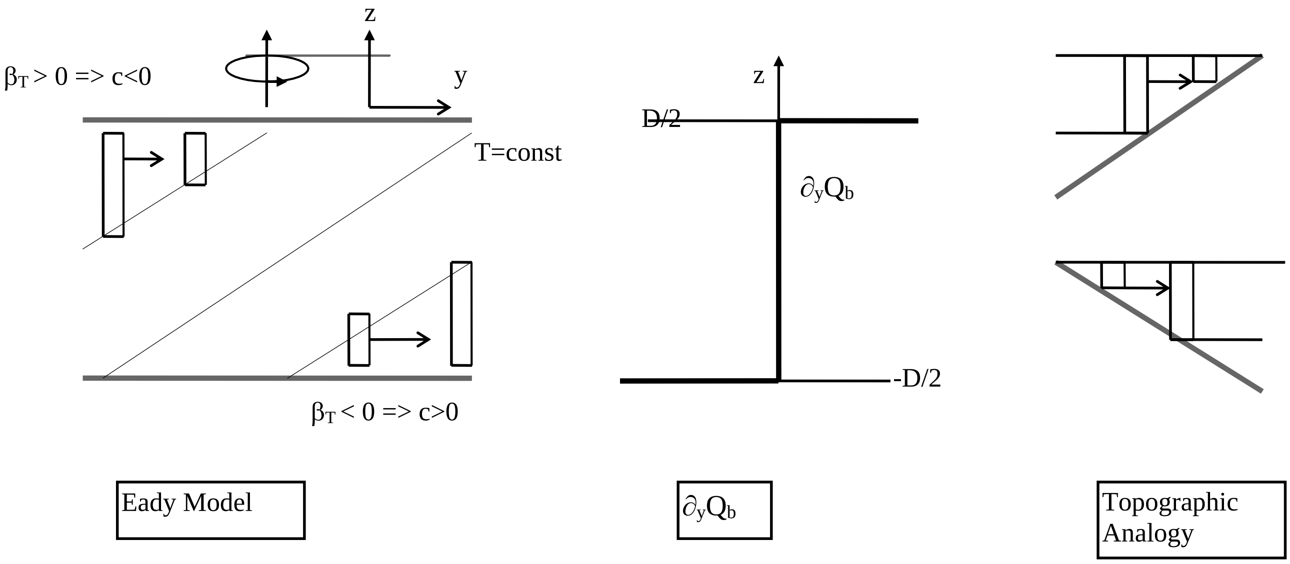


Fig.6.2 Boundary effect and “topographic” Rossby waves

3. Unstable Modes

As the wave length increases ($\mu D= L_{D1}/L$ decreases), the two modes start to coalesce and interfere such that for $L/L_{D1} <d_c$, they merge and propagate together at a phase speed equal to the middle level mean flow (Fig.6.2). The growing mode tilt with height westward, as it must be to extract APE from the mean flow (section 6.3). The growth rate maximizes for smallest possible value of l (which goes to zero for a y-unbounded system), but a finite value of k, at $k=k_m \approx 2f_0/ND = 2/L_{D1}$ for $l=0$. Thus, the scale of the fastest growing disturbance, k_m^{-1}, is at the order of the (internal) Rossby radius. Take typical atmospheric numbers $f_0=10^{-4}\ s^{-1}$, $N=10^{-2}\ s^{-1}$, $D=10\ km$, we find $k_m=10^{-6}\ m^{-1}$, which gives the wavelength $L= \pi/k_m$ about 3000 km. The maximum growth rate is about $k_m c_i \approx 0.5SDk_m$, for a wind shear of $30\ m/s$ over $10 km$ depth, $S=3*10^{-3}s^{-1}$, the growth rate is $k_m c_i \approx 1.5*10^{-5}\ s^{-1}$, or about 2 days. Both the length scale and growth rate agree well with observed cyclone development. Thus, the baroclinic instability for the first time gives a quantitative physical explanation of cyclone development. (Typical parameter in the ocean gives a spatial scale about $50\ km$, and growth time about 2 months).

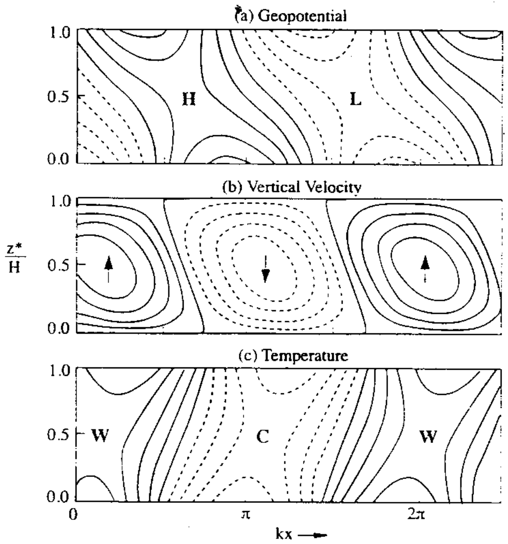


Fig. 8.10 Properties of the most unstable Eady wave. (a) Contours of perturbation geopotential height; *H* and *L* designate ridge and trough axes, respectively. (b) Contours of vertical velocity; up and down arrows designate axes of maximum upward and downward motion, respectively. (c) Contours of perturbation temperature; W and C designate axes of warmest and coldest temperatures, respectively. In all panels 1 and 1/4 wavelengths are shown for clarity.

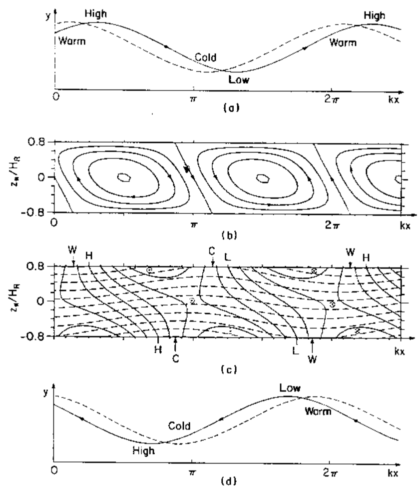


Fig. 13.4. Properties of the most unstable Eady wave, i.e., the most unstable wave in a uniform shear flow between two horizontal boundaries in a uniformly rotating environment. The solution is independent of y. (a) The pattern on the upper surface and (d) the pattern on the lower surface, the solid line being an isobar and the dashed line an isotherm. The pattern on the lower surface is very similar to that of the boundary wave shown in Fig. 13.1d, except that isotherms are now phase-shifted 21° to the east relative to the isobars. (The phase shift is exaggerated in (a) and (d) for clarity, but (b) and (c) are accurate representations.) The poleward flow is now on the warm side and the equatorward flow is on the cold side so there is net poleward heat flux. At the furthest poleward point of the isotherm, the flow is still poleward because displacements are increasing with time. The pattern at the upper surface can be obtained from that at the lower surface by symmetry. (b) The stream function for the ageostrophic flow in the x–z_* plane. Ascent is associated with "coldward" flow and descent with "warmward" flow as was found in Section 12.10. The most unstable wave has wavenumber such that there are 1.6 Rossby heights H_R between the two horizontal boundaries. (c) Contours of normal velocity v (solid) and isentropes (dashed) in the x–z_* plane. The lines marked H (high geopotential) and L (low geopotential) are zero lines for v. The points marked W (warm) and C (cold) on the boundary show where the air is warmest and coldest. At all levels, air going poleward is generally warmer than air going equatorward, so there is a net poleward heat flux. The phase lines of the v field (as for Φ') tilt westward with height, the total change in phase between the two boundaries being 90°.

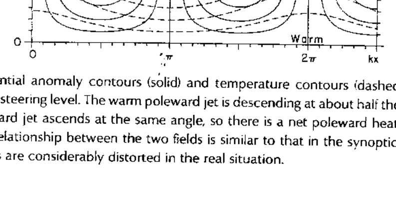


Fig. 13.5. Geopotential anomaly contours (solid) and temperature contours (dashed) for a growing square ($k = l$) Eady wave at the steering level. The warm poleward jet is descending at about half the slope of the isentropes and the cold equatorward jet ascends at the same angle, so there is a net poleward heat transfer and release of potential energy. The relationship between the two fields is similar to that in the synoptic situation shown in Fig. 12.17 although the fields are considerably distorted in the real situation.

The energetics and wave activity features of the Eady mode are consistent with our previous studies of baroclinic unstable waves. Since now $q' = 0$, we have $\nabla \cdot \boldsymbol{F} = \langle v'q' \rangle = 0$. Hence, the E-P flux must be nondivergent in the interior. Since $\partial_y \langle u'v' \rangle = 0$ (this is easy to show with the assumed form of solution), it follows that $\partial_z F_z = 0$. Thus, the vertical E-P flux, or the northward heat transport $\langle v'T' \rangle$, is constant with height. Its value can be found from the boundaries. Given

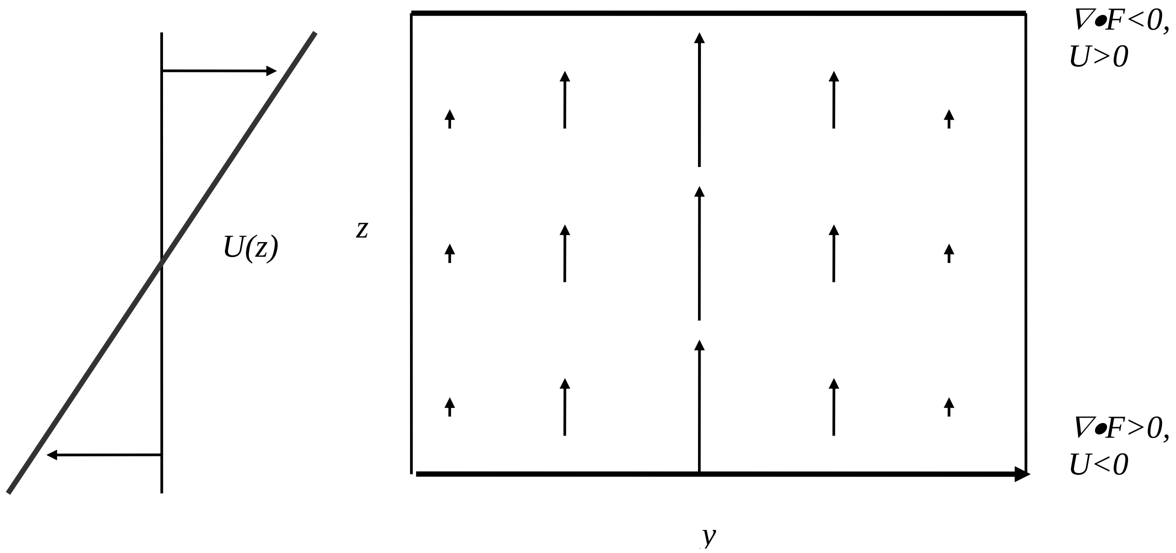
$$(\partial_t + U\partial_x)T' + v'T_y = 0, \qquad z = \pm D/2$$

and a negative T_y, it follows that the boundary heat flux

$$\langle v'T' \rangle = -\partial_t(\langle T'^2 \rangle/2)\overline{T}_y, \qquad z = \pm D/2$$

is positive, as it must be for growing modes. The $F_z = 0$ immediately beyond the boundary sheets of the generalized PV (6.5.2) because $T' = 0$ there. Thus, the lower boundary has a divergence of F_z while the upper boundary a convergence, negatively correlated with the mean wind U on the two boundaries. This gives $\iint dxdyU\nabla \cdot \boldsymbol{F} < 0$.

Therefore, energy is converted from the mean flow to the perturbation flow according to (6.3.10).



4. Charney Model

The most unrealistic feature of the Eady model is the lack of a planetary β. Charney (1947) considered the case of a beta-plane and compressible atmosphere. The upper boundary temperature gradient effect, however, is absent now. Nevertheless, since the lower boundary still provides a negative PV gradient sheets, whose sign is the opposite to the interior PV gradient $\beta>0$, baroclinic instability is still possible.

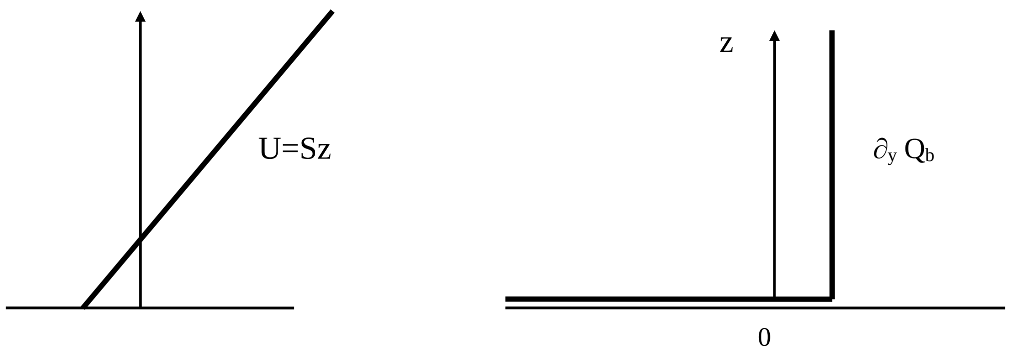


Fig.6.3 Charney Problem, mean flow and PV gradient

The scale and growth rate for the most unstable mode is similar to those of Eady's. However, the structure of the Charney's mode is very different from the Eady mode. The Charney mode decays with height and is mostly confined between the lower boundary and the critical level (where $U=c_r$). Above the critical level, the disturbance decays rapidly with height, with an almost constant phase. The E-P flux is mostly confined to the lower level near the surface, with a large heat flux. (Pedlosky, Fig.7.8.4, 5 and Gill, Fig.13.6).

Physically, the reversal of PV gradient is now at the lower boundary, it is those bottom trapped waves that can feel strongly the reversal of PV gradient. This is why now the unstable waves are bottom trapped. Furthermore, since even very small vertical scale (usually corresponding to small horizontal scale) waves can feel the reversal of PV gradient. Charney's model, unlike Eady's model, does not have shortwave cut off (ignoring friction of course). That is: no matter how small is the wave length, there can be unstable modes (Green's modes).

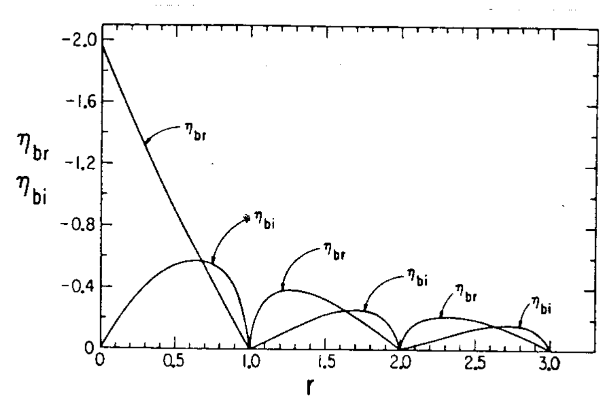


Figure 7.8.4 The real and imaginary parts of c as calculated by Kuo (1973) for the case $\delta \to 0$. In this figure $\eta_{br} = \mathrm{Re}\,\xi_0$, $\eta_{bi} = \mathrm{Im}\,\xi_0$, where $\xi_0 = -2\mu c$ where μ is the nondimensional wave number.

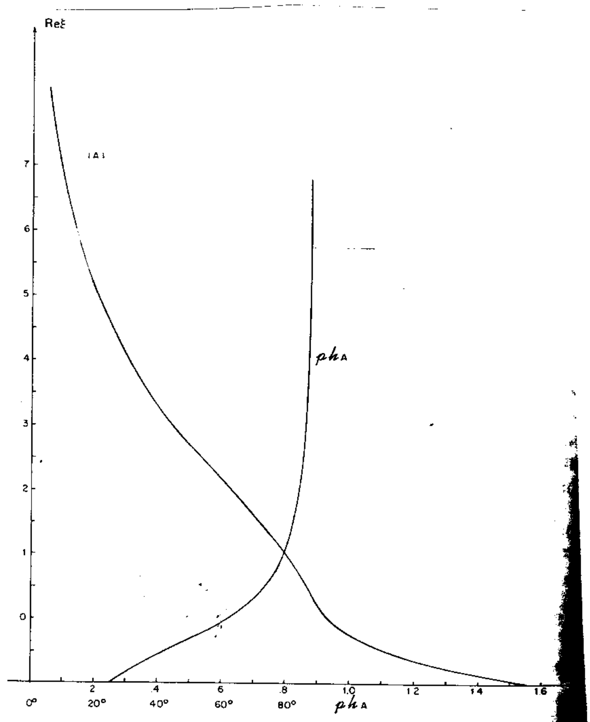


Figure 7.8.5 The amplitude and phase of the most unstable mode, i.e., at r
(Courtesy H. L. Kuo (1973).)

Fig.6.5.4 Eigenvalues of the Charney problem and the structure of the Charney mode

Sec. 6.6: Barotropic Instability*

When the PV gradient changes sign primarily through the barotropic shear term, the resulting instability is known as barotropic instability. Consider a homogeneous shallow water system with a rigid lid. The potential vorticity gradient is

$$q_y = \beta - U_{yy} \ . \tag{6.6.1}$$

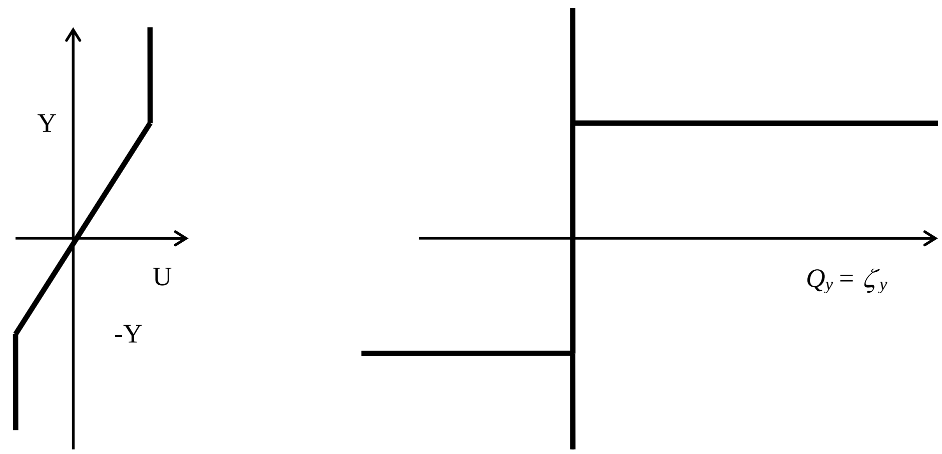
Denote the spatial scale and flow strength of the mean shear as L and U, the Charney-Stern theorem suggests that $L^2 < U/\beta$ is necessary for barotropic instability. For the midlatitude ($\beta \sim 1.5*10^{-11}\ m^{-1}\ s^{-1}$) atmosphere and ocean, $U \sim 20\ ms^{-1}$ and ~0.01.ms^{-1}, giving $L < 1100\ km$ and < $80\ km$; respectively. Therefore, barotropic instability is possible only in very narrow atmospheric or oceanic jets.

Consider a simple example where the barotropic fluid has a basic state flow

$$U = \begin{cases} SY, & y > Y \\ Sy, & -Y < y < Y \\ -SY, & y < -Y \end{cases}$$

The flow is on an f-plane ($\beta=0$, so that this example is a classical shear - inflection point - instability). Then, the mean PV gradient is zero except at $y=\pm Y$,

$$Q = S\delta(y - Y) - S\delta(y + Y) \ , \tag{6.6.1a}$$



This situation clearly has parallels with the Eady problem of baroclinic instability (Section 6.5). The general perturbation (potential) vorticity equation is:

$$(\partial_t + U\partial_x)\zeta' + \partial_x\psi' Q_y = 0 \tag{6.6.2}$$

Except at $y=\pm Y$, (6.6.1) shows $Q_y = 0$ and therefore the perturbation equation reduces to

$$(\partial_t + U\partial_x)\zeta' = 0 \ . \tag{6.6.3}$$

Therefore $\zeta' = 0$ except $y=\pm Y$. We look for solutions of the form

$$\psi' = Re\phi(y)e^{ik(x-ct)} \ ,$$

and insist that the solution is finite at $y \to \pm\infty$. Eqn. (6.6.3) therefore gives the y structure as

$$\phi = \begin{cases} Ae^{-ky}, & y > Y \\ Be^{ky} + Ce^{-ky}, & -Y < y < Y \\ De^{ky}, & y < -Y \end{cases} \tag{6.6.4a,b,c}$$

The coefficients are determined by the matching conditions across $y=\pm Y$. First, ϕ is continuous across $y=\pm Y$. This gives

$$A = Be^{2kY} + C, \quad D = B + Ce^{2kY} . \tag{6.6.5}$$

Second, $u' \sim d\phi/dy$ has a finite jump across $y=\pm Y$, whose value is determined as follows. Near $y = +Y$, the perturbation equation (6.6.2) is

$$(SY - c)\zeta' + \psi' S\delta(y - Y) = 0 . \tag{6.6.6}$$

Since $\zeta' = \partial_x v' - \partial_y u'$, the singularity in ζ' must indicate a discontinuity in u'

$$\int_{Y-}^{Y+} \zeta' dy = u'(x, Y-) - u'(x, Y+).$$

Therefore, integrating (6.6.6) in the vicinity of $Y+$ gives

$$(SY - c)[d\phi/dy(Y+) - d\phi/dy(Y-)] + S\phi(Y) = 0 . \tag{6.6.6a}$$

Similarly, at $y = -Y$,

$$(SY + c)[d\phi/dy(-Y+) - d\phi/dy(-Y-)] + S\phi(-Y) = 0 . \tag{6.6.6b}$$

Substitute (6.6.4) and (6.6.5) into (6.6.6a,b), we have

$$[S - 2k(SY - c)]Be^{kY} + SCe^{-kY} = 0 ,$$

$$SBe^{-kY} + [S - 2k(SY + c)]Ce^{kY} = 0 . \tag{6.6.7}$$

The eigenvalues are determined by setting the determinant zero as

$$4k^2c^2 = S^2[(1 - skY)^2 - e^{-4kY}] . \tag{6.6.8}$$

Very short waves are stable because the RHS is positive in the limit of large kY. Calculations show that instability occurs only for long waves of $kY < 0.639$, beyond which we have a shortwave cutoff (Fig.6.6.1). The reason of the shortwave cut-off is the same as in the Eady problem - short waves in x are of small scales in y (see the solution (6.6.4)) and therefore won't feel the sign change of the mean PV gradient across $y=\pm Y$ simultaneously. The maximum growthrate is found occur at $kY=0.398$.

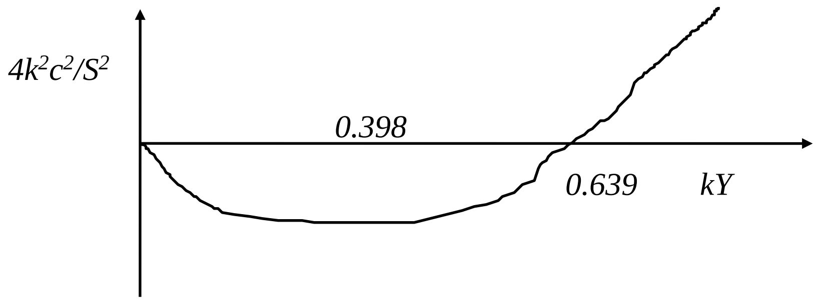


Fig.6.6.1 Eigenvalue of the barotropic instability

The structure of the growing modes (Fig.6.6.2) is again quite similar to the Eady waves. Maximum streamfunction amplitude is found at $y=\pm Y$. In $-Y<y<Y$, the mode tilts westward with y; this mean that $\langle u'v'\rangle < 0$ or, equivalently, $F_y >0$. Since $\nabla\bullet\mathbf{F}$ can be nonzero only at $y=\pm Y$, F_y is constant and positive in $-Y < y < Y$ and zero for $|y|>Y$. Therefore,

$$\int_{-\infty}^{\infty} \mathrm{U}\,\nabla\cdot\boldsymbol{F}dy < 0$$

as it must be for the mean state to loss kinetic energy . The energy source is the mean kinetic energy this time.

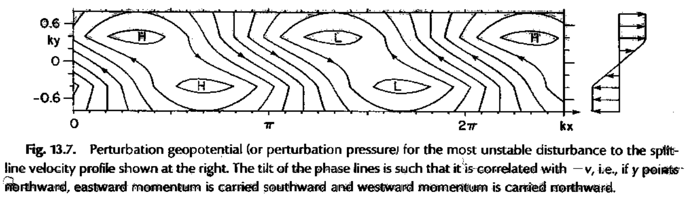


Fig.6.6.2 Eigenfunction of the barotropic instability

Questions for Chapter 6

Q6.1. Derive the 2-layer QG model from the continuously stratified QG equation as a level model, using finite difference in the vertical (assuming N=const).

Q6.2. The 1.5-layer model can be understood as the limit of a 2-layer model if you follow these steps. a) derive the 2-layer QG model with different layer depths, b) In the absence of mean flows, derive the linear Rossby waves, their eigenvalues and eigenfunctions. c) Under what conditions, the 2-layer model in (a) reduces to the 1.5 layer model. d) Which mode in (c) approaches the Rossby wave mode derived from the 1.5-layer model.

Exercises for Chapter 6

E6.1. (Baroclinic wave structure) In the 2-layer QG model, with a mean flow U, the eigenvalues are derived in (6.2.14) and the eigenfunctions can be determined from (6.2.13). Discuss the vertical structures of the wave (i.e. A_1/A_2) for each eigenmodes in the following cases. a) U=0, but $\beta \neq 0$, b) $\beta=0$, but $U \neq 0$.

E6.2. (Forced baroclinic ocean response) A linear 2-layer QG model is forced by a weak wind curl perturbation $curl\tau = Aexp[i(kx + ly - \sigma t]$. Assuming the mean state is motionless, so the forced response satisfies

$$\partial_t[\nabla^2\psi_1 - F(\psi_1 - \psi_2)] + \beta\partial_x\psi_1 = curl\boldsymbol{\tau} \ ,$$

$$\partial_t[\nabla^2\psi_2 + F(\psi_1 - \psi_2)] + \beta\partial_x\psi_2 = 0 \ .$$

Find the response to the wind forcing and discuss the response as a function of the forcing frequency σ. What happens when σ approaches zero?

E6.3. (Long baroclinic waves) On a f-plane (*β=0),* assume a constant mean flow of U (>*0*) in the upper and –U in the lower layer, the perturbation PV equations in the 2-layer model are:

$$(\partial_t + U\partial_x)\left[\nabla^2\psi'_1 - F\left(\psi'_1 - \psi'_2\right)\right] + 2FU\partial_x\psi'_1 = 0 \ ,$$

$$(\partial_t - U\partial_x)[\nabla^2\psi'_2 + F(\psi'_1 - \psi'_2)] + 2FU\partial_x\psi'_2 = 0 \ .$$

The solution can be derived in the form of $\psi'_n = Re[A_n e^{ik(x-ct)+ily}]$. In the long wave limit $K^2 = k^2 + l^2 \to 0$,

a) Prove that the eigenvalues approaches $C \to \lambda iU$, where $\lambda = \pm 1$ ($\lambda = +1$ for one mode, $\lambda = -1$ for the other mode).

b) Prove that the eigenfunction structure satisfies: $A_1/A_2 \to \lambda i$.

c) Prove that the perturbation streamfunctions can be written in the form $\psi'_2 = con(\theta)$, $\psi'_1 = con(\theta + \lambda\pi/2)$, where $\theta = kx + ly$.

d) Prove that the perturbation heat flux approaches $\langle (v'_1 + v'_2)(\psi'_1 - \psi'_2) \rangle \to \lambda k$, where $\langle\ \ \rangle$ is the zonal mean over one wave length.

e) Which is the unstable mode? What is its vertical structure (tilting west or east, by how much? What is the direction of perturbation heat flux?

f) repeat e), but for the decaying mode

E6.4. In the interior ocean (x>0), baroclinic waves in a 2-layer fluid satisfy:

$$\partial_t[\nabla^2\psi_1 - F(\psi_1 - \psi_2)] + \beta\partial_x\psi_1 = 0 ,$$
$$\partial_t[\nabla^2\psi_2 + F(\psi_1 - \psi_2)] + \beta\partial_x\psi_2 = 0 .$$

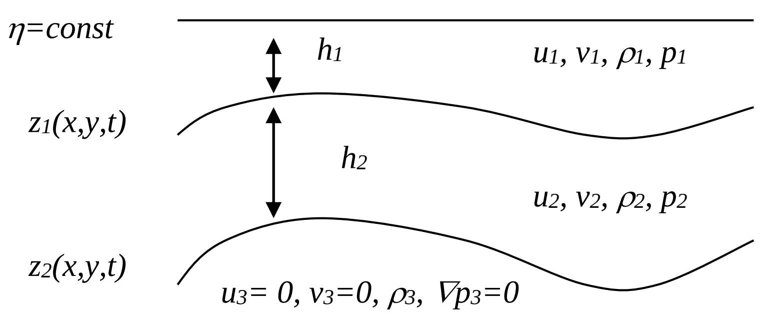
There is a wave maker on the eastern boundary of the ocean (*x=0*), which generates the perturbation of $\psi_1 = Ae^{-i\sigma t}$, $\psi_2 = 0$, at *x=0*. Find the boundary forced waves? Discuss the wave response as a function of the forcing frequency σ.

E6.5. (2.5-layer QG model) In a 2.5-layer fluid (see E1.4) with a rigid lid (such that surface elevation is negligible), using the PV approach similar to Sec. 6.2, show that the QGPV equations can be written as $[\partial_t + J(\psi_n,\ \ )]q_n = 0,\quad n = 1, 2$, and

$$q_1 = \beta y + \nabla^2\psi_1 - (\psi_1 - \psi_2) f_0^2/(g'_1 D_1),\quad q_2 = \beta y + \nabla^2\psi_2 + (\psi_1 - \psi_2) f_0^2/(g'_1 D_2) - \psi_2 f_0^2/(g'_2 D_2).$$

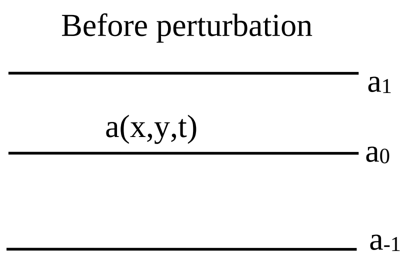
where D_1 and D_2 are the mean thickness of the two layers.

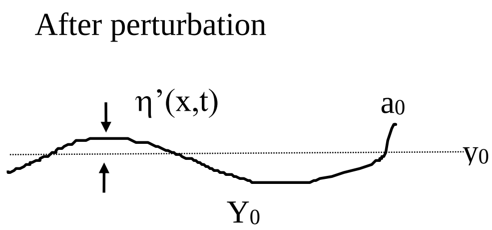
(Hint: in light of E1.4b, the perturbation interface depth anomalies z'_1 and z'_2 can be related to the QG streamfunctions as $f_0\psi_1 = -g_1 z'_1 - g_2 z'_2$ and $f_0\psi_2 = -g_2 z'_2$.)



E6.6. (2.5-layer planetary waves) Free planetary Rossby waves in the 2.5-layer model can be derived by neglecting relative vorticity in the 2.5-layer QG PV model that is derived in E6.5. Discuss the baroclinic planetary waves for the two cases, where the mean state (a) is motionless, and (b) has a mean flow of $U_1 = U_2 = U$ in the two layers. When U increases to infinity, what are the wave speeds of the two modes? Under what conditions, does the wave become unstable? For simplicity, you can assume $D_1 = D_2$ and $g'_1 = g'_2$.

E6.7. For a conservative quantity a, with $da/dt = 0$, prove that a small perturbation of a satisfies $a' = -\eta' A_y$, where η' is the disturbance distance deviation.





Hint: $a_0 \equiv a[x, Y_0(x,t)]$ gives the position of the contour a_0 as $Y_0(x,t)$. Assume the time mean position of $Y_0(x,t)$ is y_0, we have

$$a_0 \equiv a[x, Y_0(x,t)] = a(x, y_0, t) + (Y_0 - y_0)\partial_y a|_{y=y_0} + \cdots \approx a(x, y_0, t) + \eta' \partial_y A$$

where $Y_0(x,t)$ is a material surface and $\eta' = Y_0 - y_0$. Thus $a' = a(x, y_0, t) - a_0 \approx \eta' \partial_y A$.

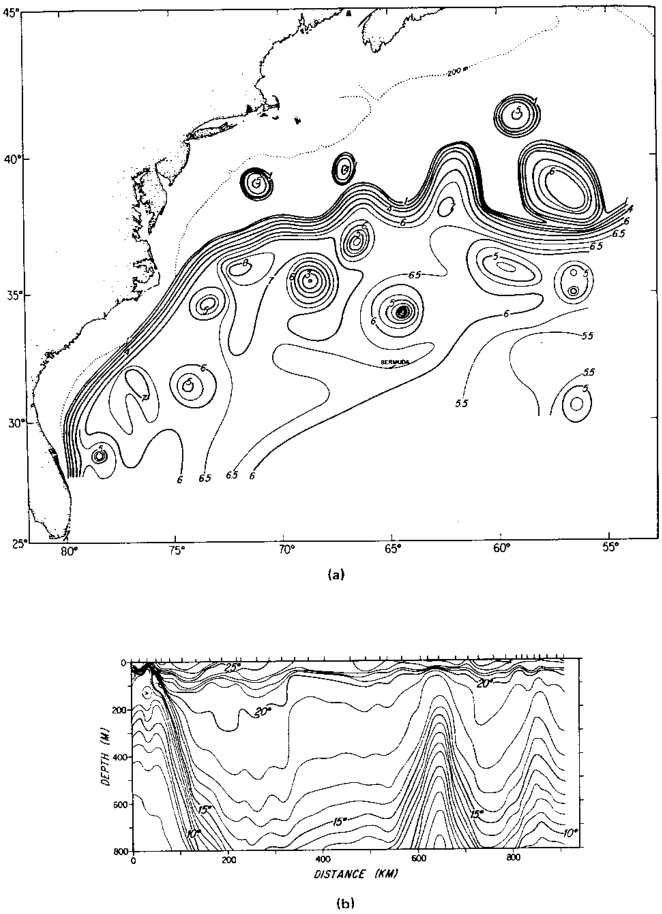
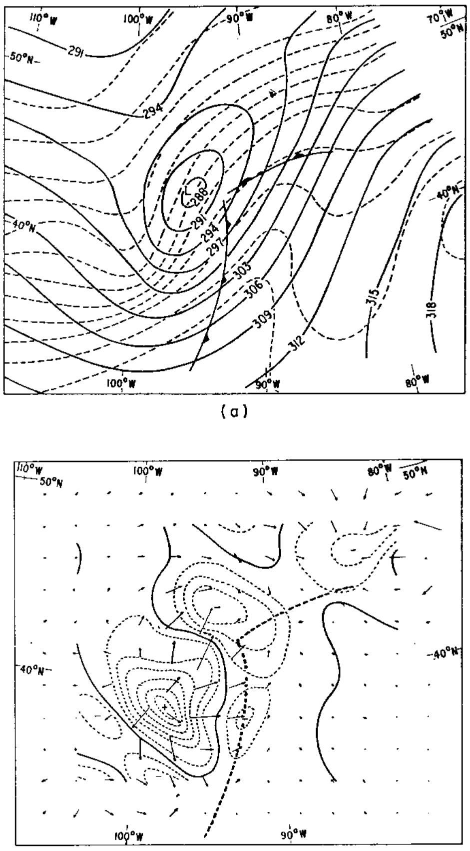


Fig. 13.8. (a) Chart of the depth, in hundreds of meters, of the 15° isothermal surface, showing the Gulf Stream, nine cyclonic rings, and three anticyclonic rings. Contours are based on data obtained between 16 March and 9 July 1975. (b) A temperature section through the Gulf Stream and two cyclonic (cold-core) rings south of the stream. The section is a "dog-leg" from 36°N, 75°W (left-hand end) to 35°N, 70°W (middle) and then to 37°N, 65°W (right-hand end). [From Richardson *et al.* (1978, Figs. 1a and 4a, Section 3).]



(a)

(b)

Fig. 12.17. (a) Analysis of the 700 mb chart for 0000 GMT on November 10, 1975. Height contours are drawn every 30 dynamic meters and temperature contours every 2°. The surface frontal analysis is indicated. (b) **Q**-vectors (arrows) and contours of their divergence (zero lines marked solid) for the situation shown in (a). [From Hoskins and Pedder (1980).]